Wavelengths Requirement for Permutation Routing in All-Optical Multistage Interconnection Networks

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Abstract *Previous studies showed that the cross-talk problem on the all-optical networks exists at both links and switches of the networks. To solve the cross-talk problem at both links and switches, one approach is to assign the wavelengths to the communication paths so that the paths which receive the same wavelength are node-disjoint. Our goal is to minimize the number of wavelengths required for permutation routings by node-disjoint paths on all-optical MINs which consists of n stages of 2×2 switches connecting $N = 2^n$ inputs and outputs. We prove that the problem of finding the minimum number of wavelengths for arbitrary partial permutation routings on the MINs is NP-complete. We show that any partial permutation routing can be realized by $2^{\lceil n/2 \rceil}$ wavelengths and there exist permutation routings that require at least $2^{\lceil n/2 \rceil}$ wavelengths. Although the general problem is NP-complete, we give an efficient algorithm for computing the minimum number of wavelengths for the class of BPC (bit permute-complement) permutations.*

Keywords Permutation routing, MINs, all-optical networks, node-disjoint paths, BPC permutations

1. Introduction

With rapid development of electro-optic technologies, the optical network is becoming a major communication infrastructure due to its high data transmission rates, low error rates, and low delay. Routing on optical networks is one of the most active research areas in communication and parallel/distributed computing [6, 15, 11, 17, 12, 21, 13]. Optical networks consist of links (optical fibers) and nodes (switches). When the electronic switches are used, the optical signals on the fiber are converted to electronic signals for processing at switches of the networks. The electro-optic conversion is a bottleneck for increasing the performance of the optical networks. In all-optical networks, optical switches are used to keep all data optical from input to output through the nodes of the networks. Since the switching is done optically and there is no optical random access memory available, there can be no data storage inside all-optical networks and circuit-switched routing mode is used for all-optical networks. To utilize the large available bandwidth in optical fibers, a single fiber can be partitioned into multiple communication channels by the wavelength-division-multiplexing (WDM) approach. To realize a routing request on the all-optical network, one needs to set up the routing paths for every pair of input and output in the routing request and assign a wavelength to each path. We assume that no wavelength conversion is available in the network. All-optical approach eliminates the electro-optic bottleneck and allows data transmission rates to reach the bounds allowed by optical devices that are significantly beyond the rates possible in electronic networks. However, this approach also introduces several challenges in the design of routing algorithms. One of the problems with the current all-optical networks is *crosstalk* at optical fibers and switches, i.e., if two signals with the same wavelength come to a same link or switch simultaneously, they are interfered with each other [8, 21]. One solution to the *crosstalk* problem at both links and switches is to set up the routing paths and assign the wavelengths to the paths so that the routing paths with the same wavelength are node-disjoint. Since the wavelengths are important resources of the all-optical networks, we need to assign as few wavelengths as possible.

In this paper, we consider the problem of finding the minimum number of wavelengths for realizing arbitrary permutation routings by node-disjoint paths on all-optical multistage interconnection networks (MINs). The MINs have been used for parallel computing systems such as IBM SP1/SP2 [18] and NEC Cenju-3 [9], and used for the internal structures of optical couplers (e.g., star couplers) [17]. We consider the MINs that consist of n stages of 2×2 switching elements connecting $N = 2^n$ inputs and outputs. The MINs have the property of *full access* that any output is reachable from any input in a single pass through the network. In addition, there is a unique path between any pair of input and output in the MINs. The class of MINs includes several important networks such as butterfly networks, Omega networks, and regular SW bayan networks with spread and fanout of 2 ($S + F = 2$).

Given an MIN, a routing request $R = \{(u, v)\}$ on the MIN is a partial permutation (resp. permutation) if each input of the MIN appears in R at most once (resp. exactly once) and each output of the MIN appears in R at most once (resp. exactly once). The permutation routing by edge-disjoint paths on the electronic MINs with circuit switch routing mode has been extensively studied in the literature [20, 19, 5, 7, 1, 16]. These works can be classified into two categories: the permutation capability of the MINs and the number of passes required to realize a permutation on the MINs. The permutation capability of an MIN is the fraction of all possible permutations that can be realized by one pass of routing, i.e., by edge-disjoint paths, in the network. For the permutation which can not be realized by one pass of routing, researchers have been interested in minimizing the number of passes for realizing the permutation by edge-disjoint paths [16, 1]. The previous edge-disjoint paths based results are not applicable to the routing by node-disjoint paths on all-optical MINs. Recently, the permutation capability of all-optical MINs based on node-disjoint paths was studied in [21]. Notice that in the all-optical MINs considered, every two inputs (resp. outputs) are connected to one switch. Since the two inputs (outputs) connected to the same switch can not be realized by node-disjoint paths, at least two wavelengths are needed for any permutation. In [21], the fraction of all possible *semi-permutations* that can be realized by node-disjoint paths on MINs was studied, where a semi-permutation is a partial permutation R with exactly one input (one output) connected to each switch appears in R.

In this paper, we minimize the number of wavelengths for permutation routings by node-disjoint paths on all-optical butterfly networks. Our results are:

1. The problem of finding the minimum number of wavelengths for arbitrary partial permutations is NP-complete.

2. Any partial permutation can be realized by $2^{\lceil n/2 \rceil}$ wavelengths and there exist permutations which require at least $2^{\lceil n/2 \rceil}$ wavelengths.

3. For any partial permutation R, if R can be realized in l ($2^{k-1} \leq l < 2^k$, $1 \leq k \leq n/2$) passes by edge-disjoint paths then R can be realized in less than $2^{k+1} + (n - 2k)l$ wavelengths by node-disjoint paths.

4. The minimum number of wavelengths for bit-permute-complement (BPC) permutations can be found in polynomial time.

5. For any BPC permutation R, if R can be realized in l passes by edge-disjoint paths then R can be realized in at most $2l$ wavelengths by node-disjoint paths.

The class of BPC permutations is an important class in parallel processing. It includes perfect shuffle, unshuffle, bit-reversal, butterfly permutations in FFT algorithms, and segment shuffles [10]. Much work has been done on the routing for BPC permutations in several interconnection networks [10, 16, 14]. Our results show that for BPC permutations on the butterfly network, routing by node-disjoint paths is as efficient (within a constant factor of 2) as routing by edge-disjoint paths. Since it has been known that several important MINs such as butterfly networks, Omega networks, and so on are topologically equivalent [20], our results on the butterfly network hold on the other equivalent networks as well.

In the next section, we give the preliminaries. The wavelengths requirement for arbitrary permutations is discussed in section 3. The routing for BPC permutations is given in section 4. The final section concludes the paper.

2. Preliminaries

The butterfly network of n stages has $N = 2^n$ inputs and outputs. The N inputs (resp. outputs) are labeled by n-bit binary numbers of $\{0, 1\}^n$. Each stage of the butterfly has $N/2$ 2×2 switches (nodes). We label the nodes at stage i ($0 \leq i \leq n - 1$) by $\langle w, i \rangle$, where w is an $(n - 1)$-bit binary number of $\{0, 1\}^{n-1}$ that denotes the *row* of the node. Two nodes $\langle w, i \rangle$ and $\langle w', i' \rangle$ are linked by an edge if and only if $i' = i + 1$ and either w and w' are identical or w and w' differ in precisely the i'th bit from left (or the i'th most significant bit). Each node $\langle w, 0 \rangle$ has two inputs with binary labels $w0$ and $w1$ and each node $\langle w, n - 1 \rangle$ has two outputs with binary labels $w0$ and $w1$. Figure 1 shows the butterfly network of three stages. We view the butterfly network as an undirected graph. A path in a graph is a sequence of edges of the form

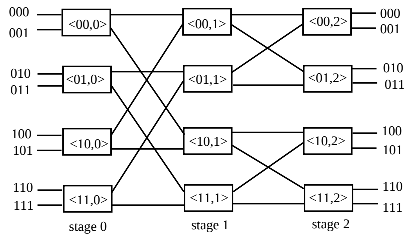

Figure 1: The butterfly network of three stages.

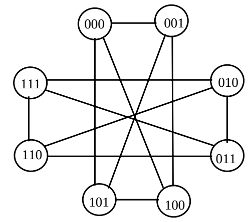

Figure 2: The node-conflict graph G_R.

$(s_1, s_2)(s_2, s_3) \ldots (s_{k-1}, s_k)$, s_i $(1 \leq i \leq k)$ are the nodes of the graph and $s_i \neq s_j$ for $i \neq j$. We sometimes denote the path from s_1 to s_k by $s_1 \rightarrow s_k$. Two paths are *node-disjoint* (resp. *edge-disjoint*) if they have no common node (resp. edge).

Given an arbitrary ordered pair (u, v) of input u with binary label $x_1 x_2 ... x_n$ and output v with binary label $y_1 y_2 ... y_n$ in the butterfly, there is a unique path from u to v: $\langle w_0, 0\rangle \rightarrow \langle w_1, 1\rangle \rightarrow ... \rightarrow \langle w_i, i\rangle \rightarrow ... \rightarrow \langle w_{n-1}, n-1\rangle$, where $w_0 = x_1 x_2 .... x_{n-1}$, $w_i = y_1 ... y_i x_{i+1} ... x_{n-1}$ $(0 < i < n-1)$, and $w_{n-1} = y_1 y_2 ... y_{n-1}$. Given two pairs (u, v) and (u', v') of inputs and outputs, where $u = x_1 x_2 ... x_n$, $v = y_1 y_2 ... y_n$, $u' = x'_1 x'_2 ... x'_n$, and $v' = y'_1 y'_2 ... y'_n$, the paths $u \rightarrow v$ and $u' \rightarrow v'$ have a common node if there is an i $(0 \leq i \leq n-1)$ such that $y_1 ... y_i x_{i+1} ... x_{n-1} = y'_1 ... y'_i x'_{i+1} ... x'_{n-1}$.

Given a routing request R, the routing paths for the pairs of inputs and outputs are uniquely defined in the butterfly. To realize R on the all-optical butterfly, we need to assign wavelengths to the routing paths in such a way that the paths with the same wavelength are node-disjoint. We say that the paths for input-output pairs (u, v) and (u', v') conflict at stage i, $0 \leq i \leq n-1$, iff the unique paths $u \rightarrow v$ and $u' \rightarrow v'$ share the same node at stage i in the network. Two paths $u \rightarrow R(u)$ and $u' \rightarrow R(u')$ conflict if they conflict at at least one stage. To describe the scenario of the node-conflicts of the paths in the butterfly, we introduce *node-conflict graph*. Given a partial permutation R on the n-stage butterfly network, the node-conflict graph $G_R(V, E)$ is a graph, where $V(G_R) = \{u | (u, v) \in R\}$ and there is an edge $(u, u') \in E(G_R)$ iff the routing paths for $(u, R(u))$ and $(u', R(u'))$ conflict in the butterfly. Figure 2 gives the node-conflict graph G_R for the permutation $R = \{(000, 100), (001, 110), (010, 001), (011, 000), (100, 101), (101, 111), (110, 010), (111, 011)\}$.
The node-conflict graph given above is similar to the *conflict graph* introduced in [16] that describes the edge-conflicts of the paths. In what follows, we call the later *edge-conflict graph*. For a partial permutation R, we often denote the corresponding output of u in R by $R(u)$.

A k-coloring of a graph G is an assignment of k distinct colors to the nodes of G such that every pair of adjacent nodes are assigned different colors [2]. A graph G that has a k-coloring is said *k-colorable*. The minimum number k that G is k-colorable is called *chromatic number* of G. It was shown in [16] that the minimum number of passes for a permutation R by edge-disjoint paths is the chromatic number of the edge-conflict graph of R. We will show in the next section that to find the minimum number of wavelengths for R by node-disjoint paths on the butterfly is equivalent to find the chromatic number of the node-conflict graph G_R.

3. Wavelengths requirement for arbitrary permutations

Theorem 1 *A partial permutation R can be routed in k wavelengths by node-disjoint paths on the butterfly network iff the node-conflict graph G_R is k-colorable.*

Proof: Assume that there is a k-coloring function $f : V(G_R) \rightarrow \{1, 2, ..., k\}$ on the nodes of $G_R(V, E)$ such that $f(u) \neq f(u')$ if $(u, u') \in E(G_R)$. For each pair $(u, R(u))$ in R, we assign the wavelength $f(u)$ to the path $u \rightarrow R(u)$. Then for any two routing paths that share a common node in the butterfly, the wavelengths assigned to the paths are different. Therefore, R can be realized by k wavelengths.

Assume that R can be realized by k wavelengths of $\{1, 2, ..., k\}$. We color the node u of node-conflict graph G_R with the color corresponding to the wavelength assigned to the path $u \rightarrow R(u)$. Then, the nodes of $G_R(V, E)$ have been colored in such a way that for any $(u, u') \in E(G_R)$, u and u' have different colors. $\square$

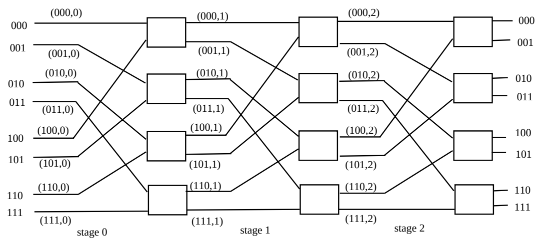

Figure 3: The Omega network of three stages.

Since the problem of finding the chromatic number of a graph is NP-complete [4], the above theorem suggests that we may not be able to find the minimum number of wavelengths for permutation routings efficiently. We prove that the problem of finding the minimum number of wavelengths for arbitrary partial permutations by node-disjoint paths on the butterfly is NP-complete. Our proof is based on the work of Bernhard and Rosenkrantz who proved that finding the minimum number of passes for routing partial permutations by edge-disjoint paths on the Omega network is NP-complete [3]. The proof of Bernhard and Rosenkrantz is to reduce the k-colorability problem of graphs with bounded degree (the problem is still NP-complete though) to the problem of finding the minimum number of passes for routing a partial permutation by edge-disjoint paths on an Omega network. More specifically, given a graph $G(V, E)$ with bounded node-degree, an Omega network and a partial permutation R on the Omega network are constructed in polynomial time in $|V(G)|$ such that G is k-colorable iff R can be routed in k passes by edge-disjoint paths on the Omega network (the edge-conflict graph of R is k-colorable).

To state our NP-complete result, we first give a brief review on Omega networks. The Omega network of n stages has $N = 2^n$ inputs and outputs. The inputs and outputs are labeled by binary numbers of $\{0,1\}^n$. The n stages of the Omega network are identical, each of them has N edges and $N/2$ 2×2 nodes. The edges in each stage implements a *perfect shuffle* connection on binary labels of $\{0,1\}^n$, i.e., $x_1x_2...x_n$ is connected to $x_2x_3...x_nx_1$. We label edges at stage i $(0 \leq i \leq n-1)$ by (e, i), where e is a binary number of $\{0,1\}^n$. Figure 3 gives an Omega network of three stages. For any input and output pair (u, v) with $u = x_1x_2...x_n$ and $v = y_1y_2...y_n$, there is a unique path $u \to v$ in the Omega network that consists of edges (e_i, i), $0 \leq i \leq n-1$, where $e_i = x_{i+1}...x_ny_1...y_i$.

Theorem 2 *Determining whether a partial permutation R can be routed in k wavelengths by node-disjoint paths on the butterfly network is NP-complete.*

Proof: It was shown in [3] that for an arbitrary graph $G(V, E)$ with bounded node-degree, an Omega network and a partial permutation R on the Omega network can be constructed in polynomial time in $|V(G)|$ such that $G(V, E)$ is k-colorable iff the edge-conflict graph of R is k-colorable. Assume that the constructed Omega network is of stage n. For the constructed partial permutation R, let $R' = \{(x, y) | x = x_1x_2...x_n0, y = y_1y_2...y_n0, (x_1x_2...x_n, y_1y_2...y_n) \in R\}$ be the routing request on the $(n+1)$-stage butterfly network. Then R' is a partial permutation. We prove that the node-conflict graph $G_{R'}$ of R' on the $(n+1)$-satge butterfly network is isomorphic to the edge-conflict graph of R on the n-stage Omega network. From this and Theorem 1, graph $G(V, E)$ is k colorable iff the partial permutation R' can be routed in k wavelengths by node-disjoint paths on the butterfly network, and the theorem holds.

Let $(u, R(u))$ and $(u', R(u'))$ be two input-output pairs in R on the n-stage Omega network with $u = x_1x_2...x_n$, $R(u) = y_1y_2...y_n$, $u' = x'_1x'_2...x'_n$, and $R(u') = y'_1y'_2...y'_n$. The path $u \to R(u)$ consists of edges (e_i, i), $0 \leq i \leq n-1$, where $e_i = x_{i+1}...x_ny_1...y_i$. Similarly, path $u' \to R(u')$ consists of edges (e'_i, i), $0 \leq i \leq n-1$, $e'_i = x'_{i+1}...x'_ny'_1...y'_i$. Paths $u \to R(u)$ and $u' \to R(u')$ are edge-disjoint iff for every i with $0 \leq i \leq n-1$, $e_i \neq e'_i$.

Now, consider the input-output pairs $(x, R'(x))$ and $(x', R'(x'))$ of R' on the $(n+1)$-stage butterfly network with $x = x_1x_2...x_n0$, $R'(x) = y_1y_2...y_n0$, $x' = x'_1x'_2...x'_n0$, and $R'(x') = y'_1y'_2...y'_n0$. The path $x \to R'(x)$ passes through the nodes $\langle w_i, i \rangle$, $0 \leq i \leq n$, $w_i = y_1...y_ix_{i+1}...x_n$. Similarly, path $x' \to R'(x')$ passes through the nodes $\langle w'_i, i \rangle$, $0 \leq i \leq n$, $w'_i = y'_1...y'_ix'_{i+1}...x'_n$. Obviously paths $u \to R(u)$ and $u' \to R(u')$ share a common edge (e_i, i) $(e_i = x_{i+1}...x_ny_1...y_i)$ at stage i $(0 \leq i \leq n-1)$ of the Omega network iff paths $x \to R'(x)$ and $x' \to R'(x')$ share a common node $\langle w_i, i \rangle$ $(w_i = y_1...y_ix_{i+1}...x_n)$ at stage i $(0 \leq i \leq n-1)$ of the butterfly network. Paths $x \to R'(x)$ and $x' \to R'(x')$ have the nodes $\langle w_n, n \rangle$ and $\langle w'_n, n \rangle$, respectively, at stage n of the butterfly network, where $w_n = y_1y_2...y_n$ and $w'_n = y'_1y'_2...y'_n$. Since R is a partial permutation, $R(u) = y_1y_2...y_n \neq R(u') = y'_1y'_2...y'_n$ and paths $x \to R'(x)$ and $x' \to R'(x')$ do not share any node at stage n of the butterfly network. Thus, the node-conflict graph $G_{R'}$ on the $(n+1)$-stage butterfly network is isomorphic to the edge-conflict graph of R on the n-stage Omega network. $\Box$.

Now, we derive upper and lower bounds on the minimum number of wavelengths for a partial permutation on the butterfly network. A rough estimation can be made by Theorem 1. For the node-conflict graph G_R, let $\rho(G_R)$ be the size of the largest clique in G_R, $\Delta(G_R)$ be the maximum node-degree of G_R, and $\lambda(G_R)$ be the chromatic number of G_R. Then $\rho(G_R) \leq \lambda(G_R) \leq \Delta(G_R)+1$.

The following theorems give more precise upper and lower bounds.

Theorem 3 *Any partial permutation R can be routed in $2^{\lceil n/2 \rceil}$ wavelengths by node-disjoint paths on the n-stage butterfly network.*

Proof: For any path $u \to R(u)$ in the butterfly, the prefix segment with length k of $u \to R(u)$ is the subpath of $u \to R(u)$ from stage 0 to stage $k-1$. The suffix segment with length k of $u \to R(u)$ is the subpath of $u \to R(u)$ from stage $n-k-1$ to stage $n-1$. Let P_k and S_k be the set of prefix segments and the set of suffix segments with length k of all routing paths for R, respectively. We claim that for $1 \leq k \leq \lceil n/2 \rceil$, the routing paths for R can be colored by at most 2^k colors such that the segments of $P_k \cup S_k$ from different paths with the same color are node-disjoint. The claim is proved by induction on k.

For $k=1$, we construct a bipartite graph between the set W_0 of nodes at stage 0 and the set W_{n-1} of nodes at stage $n-1$. There is an edge between a node $w \in W_0$ and a node $w' \in W_{n-1}$ iff there is a routing path for R that passes through w and w'. Obviously, the bipartite graph has node-degree at most two. Then the edges of the bipartite graph can be partitioned into two matchings (see, for example, Berg[[2], Chap. 12 Theorem 2]). We use two colors for the edges in the two matchings, one color for one matching. Then the paths for R can be colored by two colors such that the segments of $P_1 \cup S_1$ from different paths with the same color are node-disjoint. Assume that the claim is true for $k-1 \geq 1$ and we prove it for k. We partition the paths for R into 2^{k-1} subsets such that the paths in the same subset has the same color. For each subset of paths, we construct a bipartite graph between the set W_{k-1} of nodes at stage $k-1$ and the set W_{n-k-1} of nodes at stage $n-k-1$. There is an edge between a node $w \in W_{k-1}$ and a node $w' \in W_{n-k-1}$ iff there is a path of the subset that passes through w and w'. The bipartite graph has node-degree at most two and therefore, the paths of the subset can be colored by two colors such that the segments of length k from different paths with the same color are node-disjoint. Since there are 2^{k-1} such subsets, $2 \times 2^{k-1} = 2^k$ colors are enough to make the claim true.

From the claim and taking $k = \lceil n/2 \rceil$, the theorem holds. $\Box$

Theorem 4 *There exists a permutation R which requires at least $2^{\lceil n/2 \rceil}$ wavelengths for node-disjoint routing on the n-stage butterfly network.*

Proof: We consider the permutation $R = \{(u,v) | u = x_1 ... x_{\lceil n/2 \rceil - 1} x_{\lceil n/2 \rceil} ... x_n \in \{0,1\}^n, v = x_{\lceil n/2 \rceil} ... x_n x_1 ... x_{\lceil n/2 \rceil - 1}\}$. Then the $2^{\lceil n/2 \rceil}$ paths pass through node $\langle w, \lfloor n/2 \rfloor \rangle$, where $w = x_{\lceil n/2 \rceil} ... x_n x_{\lceil n/2 \rceil + 1} ... x_n$. From this, it needs at least $2^{\lceil n/2 \rceil}$ wavelengths to realize R in the n-stage butterfly network. $\Box$

Notice that the number of wavelengths for R by node-disjoint paths is at least the number of passes for R by edge-disjoint paths on the butterfly. Previous work showed that a partial permutation R on the butterfly can be routed in $2^{\lfloor n/2 \rfloor}$ passes by edge-disjoint paths and there is a permutation R that requires $2^{\lfloor n/2 \rfloor}$ passes [1]. For a partial permutation R, let r_p and r_w be the minimum number of passes for routing R by edge-disjoint paths and the minimum number of wavelengths for routing R by node-disjoint paths on the butterfly network, respectively. Obviously $r_p \leq r_w$. An interesting problem is to find an upper bound on r_w with respect to on r_p. A straightforward upper bound is that $r_w \leq (n+1) r_p$: Partition R into r_p subsets such that the routing paths for each subset are edge-disjoint. The node-conflict graph for the paths in each subset has node-degree at most n. Therefore, the paths for each subset can be colored by at most $n+1$ colors such that the paths receiving the same color are node-disjoint. We now give a better upper bound on r_w.

Theorem 5 *For a partial permutation R with $2^{k-1} \leq r_p < 2^k$ $(1 \leq k \leq n/2)$, $r_w < 2^{k+1} + (n-2k) r_p$.*

Proof: We prove the theorem by counting the node-degree of the node-conflict graph G_R. Let $(u, R(u))$ be any pair of input and output in R. Assume that path $u \to R(u)$ passes through node $\langle w_i, i \rangle$. Then the paths which intersect $u \to R(u)$ at node $\langle w_i, i \rangle$ come into $\langle w_i, i \rangle$ from two input edges of $\langle w_i, i \rangle$. For the paths on the same input edge of $u \to R(u)$, they intersect $u \to R(u)$ at stage $i-1$ as well. The number of paths from the other input edge is at most 2^i for $0 \le i \le k-1$ and $n-k \le i \le n-1$, and at most r_p for $k \le i \le n-k-1$. Therefore, path $u \to R(u)$ intersects at most $\sum_{i=0}^{k-1} 2^i = 2^k - 1$ paths at stages from 0 to $k-1$, at most 2^k-1 paths at stages from $n-k$ to $n-1$, and at most $(n-2k)r_p$ paths at stages from k to $n-k-1$. Hence, the degree $\Delta(G_R)$ is bounded by

$$2 \times (2^k - 1) + (n-2k)r_p = 2^{k+1} + (n-2k)r_p - 2.$$

From this and $r_w \le \Delta(G_R) + 1$, the theorem holds. $\square$

4. Wavelengths requirement for BPC permutations

In this section, we show how to realize arbitrary permutations on the BPC class on all-optical butterfly networks. We first derive some important properties of the node-conflict graph produced by a BPC permutation on the butterfly network. Then we give an algorithm which finds the minimum number of wavelengths for any permutation on the BPC class.

A permutation $R = \{(u,v)\}$ of $N = 2^n$ inputs and outputs is called a BPC permutation if for input $u = x_1x_2...x_n$, $R(x_1x_2\dots x_n) = y_{i_1}y_{i_2}\dots y_{i_n}$, where $\{i_1, i_2, \dots, i_n\}$ is a permutation of $\{1, 2, \dots, n\}$, and $y_j = x_j$ or $\bar{x}_j$ for $1 \le j \le n$. R is called a BP permutation if $y_j = x_j$, $1 \le j \le n$ (i.e., $R(x_1x_2...x_n) = x_{i_1}x_{i_2}...x_{i_n}$). Many frequently used permutations in parallel processing such as perfect-shuffle, unshuffle, bit-reversal, butterfly permutations used in FFT algorithms, and segment shuffles belong to the BPC class of permutations.

Let R be a BPC permutation defined by the permutation $\{i_1, i_2, ..., i_n\}$ of $\{1, 2, ..., n\}$. The set of *free-bits* of R at stage j, $0 \le j \le n-1$, is defined as $S_{R,j} = \{i | i \in \{1, 2, ..., j, n\} \setminus I_j\}$, where $I_0 = \emptyset$ and $I_j = \{i_1, ..., i_j\}$. For two inputs $u = x_1x_2...x_n$ and $u' = y_1y_2...y_n$, it is easy to check that two paths $u \to R(u)$ and $u' \to R(u')$ conflict at stage j $(0 \le j \le n-1)$ iff $x_i = y_i$ for all $i \in \{1, 2, ..., n\} \setminus S_{R,j}$. Therefore, given a BPC permutation R, the node-conflict graph G_R can be decided by the free-bits sets $S_{R,0}, ..., S_{R,n-1}$.

Lemma 6 *The node-conflict graphs of the BPC permutation R and the BP permutation R' defined by the same permutation $\{i_1, i_2, ..., i_n\}$ of $\{1, 2, ..., n\}$ are identical.*

Proof: The Lemma follows from the observation that free-bits sets $S_{R,j} = S_{R',j}$ for $0 \le j \le n-1$. $\square$

From the above lemma, to realize a BPC permutation on the all-optical butterfly network, it is sufficient to consider the routing of the corresponding BP permutation on the network. In the rest of this paper, we consider only BP permutations.

For a BP permutation R, since any two paths $u \to R(u)$ and $u' \to R(u')$ conflict at stage j iff the binary labels of u and u' are identical on the bits in $\{1, \dots, n\} \setminus S_{R,j}$, the nodes whose binary labels are identical on the bits in $\{1, \dots, n\} \setminus S_{R,j}$ form a clique in the node-conflict graph G_R. Since there are $2^{|S_{R,j}|}$ nodes whose binary labels are identical on the bits in $\{1, \dots, n\} \setminus S_{R,j}$, the size of the clique is $2^{|S_{R,j}|}$ and the number of cliques in G_R defined by $S_{R,j}$ is $2^{n-|S_{R,j}|}$. The class $\mathcal{S}_R \subseteq \{S_{R,0}, ..., S_{R,n-1}\}$ is called a *character class* of R if $\mathcal{S}_R$ is the maximum subclass of $\{S_{R,0}, ..., S_{R,n-1}\}$ such that for any $S_{R,i}, S_{R,j} \in \mathcal{S}_R$, $S_{R,i} \not\subseteq S_{R,j}$ and $S_{R,j} \not\subseteq S_{R,i}$. Notice that if $S_{R,j} \subseteq S_{R,i}$, each clique defined by $S_{R,j}$ is a subgraph of a clique defined by $S_{R,i}$. Therefore, the the node-conflict graph G_R is defined by the character class $\mathcal{S}_R$. From the above discussion, we have the following lemma.

Lemma 7 *The node-conflict graph G_R of a BP permutation R is regular, and the node degree $\Delta(G_R)$ of G_R is $\sum_{S_{R,j} \in \mathcal{S}_R} (2^{|S_{R,j}|} - 1)$.*

Since the maximal clique in G_R is of size 2^k, $k = \max_{0 \le j \le n-1} |S_{R,j}|$, at least 2^k colors are needed for coloring G_R. We state this in the following lemma.

Lemma 8 *Let G_R be the node-conflict graph of a BP permutation R. Then, the chromatic number of G_R is at least 2^k, $k = \max_{0 \le j \le n-1} |S_{R,j}|$.*

Next, we develop an algorithm for finding an optimal coloring of the node-conflict graph G_R for an arbitrary BP permutation R. The algorithm produces a coloring of G_R with 2^k colors, where $k = \max_{0 \le j \le n-1} |S_{R,j}|$. Such a coloring is optimal since G_R contains cliques of size 2^k.

The input of the algorithm is the character class $\mathcal{S}_R$ which defines the node-conflict graph G_R, and the output of the algorithm is an optimal coloring function f_R on the nodes of G_R. The algorithm works as follows. We assign each bit a of every free-bits set $S_{R,j} \in \mathcal{S}_R$ a distinct weight $w(a) \in \{0, 1, \dots, k-1\}$, where $k = \max_{S_{R,j} \in \mathcal{S}_R} |S_{R,j}|$. The assignment is done in the increasing order on the indices j of $S_{R,j} \in \mathcal{S}_R$. Without loss of generality, assume that $\mathcal{S}_R = \{S_{R,0}, S_{R,1}, ..., S_{R,l}\}$. We assign arbitrarily a distinct weight for each bit of $S_{R,0}$.

For each free-bits set $S_{R,j}$ $(1 \le j \le l)$, let $S^1_{R,j} = S_{R,j} \cap (\cup_{0\le i\le j-1} S_{R,j})$ and $S^2_{R,j} = S_{R,j} \setminus S^1_{R,j}$. The weights for the bits of $S^1_{R,1}$ have been assigned in the previous rounds. We only need to assign the weights for the bits of $S^2_{R,j}$. To do so, we first remove the weights that have been assigned to $S^1_{R,j}$ from $\{0,1,...,k-1\}$. Since $|S_{R,j}| \le k$, there are at least $|S^2_{R,j}|$ weights which have not been used. Then we assign each bit of $S^2_{R,j}$ a distinct weight from the unused weights. After every bit in $S_{R,0}, S_{R,1}, ..., S_{R,j}$ has been assigned a weight, the coloring function $f_R : \{0,1,\ldots,N-1\} \to \{0,1,\ldots,2^k-1\}$ can be constructed.

The algorithm is shown in Figure 4. It produces the coloring function f_R which completely specifies how to realize the permutation R on the all-optical butterfly network with 2^k wavelengths. For constructing function f_R, we first consider the edge (u,u') in a clique C_j in G_R, where $u = x_1x_2\ldots x_n$, $u' = y_1y_2\ldots y_n$, and C_j is a clique generated by $S_{R,j}$. Since $x_i = y_i$ for all $i \in \{1,2,\ldots,n\} \setminus S_{R,j}$ and each $a \in S_{R,j}$ has been assigned a distinct weight $w(a)$, in order to guarantee $f_R(u) \ne f_R(u')$, it is natural to define $f_R(u) = \sum_{a\in S_{R,j}} x_a 2^{w(a)} + h(u)$, where $h(u)$ is any function determined by x_i, $i \in \{1,2,\ldots,n\} \setminus S_{R,j}$, only. For $(u,u') \in C_j$, since $x_i = y_i$ for $i \in \{1,2,...,n\} \setminus S_{R,j}$, $h(u) = h(u')$. Thus, from $\sum_{a\in S_{R,j}} x_a 2^{w(a)} \ne \sum_{a\in S_{R,j}} y_a 2^{w(a)}$ $f_R(u) \ne f_R(u')$ if $(u,u') \in C_j$. Considering any $(u,u') \in G_R$, by the extension of the observation above, $f_R(u)$ should be of form $\sum_{a\in \cup_{0\le j\le l} S_{R,j}} x_a 2^{w(a)}$. However, this is not a legal assignment since $f_R(u) \ge 2^k$ for some u. By defining $f_R(u) := \sum_{a\in \cup_{0\le j\le l} S_{R,j}} x_a 2^{w(a)} \bmod 2^k$, we show in Lemma 9 that the coloring function is correct.

We now show the correctness of algorithm COLORING in Lemma 9.

Lemma 9 *Given a BP permutation R on the butterfly network, algorithm COLORING gives a coloring function f_R such that for any edge $(u,u') \in G_R$, $f_R(u) \ne f_R(u')$.*

Proof: Obviously, every bit $a \in S_{R,j}$ $(0 \le j \le l)$ is assigned a weight $w(a)$ in $\{0,1,\ldots,k-1\}$ such that for distinct a, $a' \in S_{R,j}$, $w(a) \ne w(a')$. Let $u = x_1x_2...x_n$ and $u' = y_1y_2...y_n$ be two inputs. If $(u,u') \in G_R$ then there exists a j $(0 \le j \le l)$ such that $x_i = y_i$ for all $i \in \{1,2,...,n\} \setminus S_{R,j}$. From this, $\sum_{a\notin S_{R,j}} x_a 2^{w(a)} = \sum_{a\notin S_{R,j}} y_a 2^{w(a)}$. Let $g(u) = \sum_{a\in S_{R,j}} x_a 2^{w(a)}$ and $g(u') = \sum_{a\in S_{R,j}} y_a 2^{w(a)}$. Since $\sum_{a\in S_{R,j}} 2^{w(a)} \le \sum_{0\le i\le k-1} 2^i < 2^k$ and $\exists a \in S_{R,j}$ s.t. $x_a \ne y_a$, we have $g(u) \ne g(u')$ and $|g(u) - g(u')| < 2^k$, where $k = \max_{0\le j\le l} |S_{R,j}|$. Therefore, $g(u) \not\equiv g(u') \pmod{2^k}$. Thus, we conclude $f_R(u) \ne f_R(u')$. □

Algorithm COLORING(S_R, f_R)
Input: Character class $\mathcal{S}_R = \{S_{R,0}, \ldots, S_{R,l}\}$.
Output: Function $f_R : \{0,1,\ldots,N-1\} \to \{0,1,\ldots,2^k-1\}$, where $N = 2^n$ and $k = \max_{1\le j\le l} |S_{R,j}|$.
begin
  for $j = 0$ **to** l **do**
    $\{S^1_{R,j} := S_{R,j} \cap (\cup_{0\le i\le j-1} S_{R,i})$;
    $S^2_{R,j} := S_{R,j} \setminus S^1_{R,j}$;
    $W := \{0,1,\ldots,k-1\} \setminus \{w(a) |\ a \in S^1_{R,j}\}$;
    for $i = 1$ **to** $|S^2_{R,j}|$ **do**
      {find the minimum bit $a \in S^2_{R,j}$;
      find the minimum weight $w \in W$;
      assign a the weight w;
      $S^2_{R,j} := S^2_{R,j} \setminus \{a\}$; $W := W \setminus \{w\}$; }
    }
  for each $u = x_1 \ldots x_n \in \{0,1\}^n$ **do**
    $f_R(u) := \sum_{a\in \cup_{0\le j\le l} S_{R,j}} x_a 2^{w(a)} \bmod 2^k$;
  return (f_R);
end.

Figure 4: The algorithm for an optimal coloring of G_R.

Theorem 10 *The algorithm COLORING realizes a BP permutation with the minimum number of wavelengths by node-disjoint paths on an n-stage butterfly network in polynomial time in $N = 2^n$.*

Proof: The least upper bound on the number of wavelengths to perform a BP permutation R equals to the chromatic number of the node-conflict graph G_R, and this number is at least 2^k, where $k = \max_{0\le j\le n-1} |S_{R,j}|$. Therefore, the coloring of G_R with 2^k colors given by the algorithm BP_BF_COLORING, which is equal to realizing R on all-optical butterfly network with 2^k wavelengths, is optimal. Obviously, the algorithm runs in polynomial time in $N = 2^n$. □

For a BPC permutation R, let r_p and r_w be the minimum number of passes for routing R by edge-disjoint paths and the minimum number of wavelengths for routing R by node-disjoint paths on the butterfly network, respectively. Obviously $r_p \le r_w$. On the other hand, from Lemmas 8 and 9, $r_w = 2^k$ and there are 2^k paths conflict at a particular node.

From this, at least $r_w/2$ paths conflict at one of the two input edges of that node. Therefore, we have the following result.

Theorem 11 $r_p \leq r_w \leq 2r_p$.

5. Concluding remarks

Node-disjoint paths routing is an approach to solve the cross-talk problem on both links and switches of all-optical networks. Our results suggest that the required node-disjointness on all-optical MINs may not be a bottleneck compared with their electronic counterparts. An intersting open problem is to find a tight bound on the number of wavelengths for any partial permutation by node-disjoint paths with respect to the minimum number of passes for the same permutation by edge-disjoint paths on the MINs. The following problems are worth further work as well: Heuristic algorithms for finding the wavelength assignment for arbitrary permutations on all-optical MINs and efficient algorithms for finding the wavelength assignment for permutations on extra-stage all-optical MINs.

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References

[1] D.P. Agrawal. Graph theoretical analysis and design of multistage interconnection networks. *IEEE Trans. on Computers*, C-32(7):637–648, 1983.

[2] C. Berge. *Graphs and Hypergraphs*. North-Holland, 1973.

[3] P.J. Bernhard and D.J. Rosenkrantz. Partitioning message patterns for bundled omega networks. *IEEE Trans. on Parallel and Distributed Systems*, 5(4):353–363, 1994.

[4] M.R. Garey and D.S. Johnson. *Computers and Intractability, a Guide to the Theory of NP-Completeness*. Freeman, New York, 1979.

[5] I. Gazit and M. Malek. On the number of permutations performable by extra-stage multistage interconnection networks. *IEEE Trans. on Computers*, C-38(2):297–302, 1989.

[6] P.E. Green. *Fiber Optic Networks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[7] S.T. Huang and S.K. Tripathi. Finite state model and compatibility theory: new analysis tools for permutation networks. *IEEE Trans. on Computers*, C-35(7):591–601, 1986.

[8] D.K. Hunter and I. Andonovic. Guided wave optical switch architectures. *International Jour. of Optoelecttronics*, 9(6):477–487, 1994.

[9] N. Koike. Nec cenju-3: a microprocessor-based parallel computer. In *Proc. of the International Parallel Processing Symposium*, pages 396–401, 1994.

[10] J. Lanfant. Parallel permutations of data: a benes network control algorithm for frequently used permutations. *IEEE Trans. on Computers*, C-27(7):637–647, 1978.

[11] M. Milail, C. Kaklamanis, and S. Rao. Efficient access to optical bandwidth. In *Proc. of the 36th Annual Symposium on Foundations of Computer Science (FOCS'95)*, pages 548–557, 1995.

[12] A. Mokhtar and M. Azizo glu. Adaptive wavelength routing in all-optical networks. *IEEE/ACM Trans. on Networking*, 6(2):197–206, 1998.

[13] L. Narayanan, J. Opatrny, and D. Sotteau. All-to-all optical routing in optimal chrodal rings of degree four. In *Proc. of the 10th Annual ACM-SIAM Symposium on Discrete Algorithms SODA'99*, pages 695–703, 1999.

[14] D. Nassimi and S. Sahni. Optimal bpc permutations on a cube connected simd computer. *IEEE Trans. on Computers*, C-31(4):338–341, 1982.

[15] R.K. Pankaj and R.G. Gallager. Wavelength requirements of all-optical networks. *IEEE/ACM Trans. on Networking*, 3(3):269–280, 1995.

[16] C.S. Raghavendra and A. Varma. Fault-tolerant multiprocessors with redundant-path interconnection networks. *IEEE Trans. on Computers*, C-35(4):307–316, 1986.

[17] R. Ramaswami and K.N. Sivarajan. *Optical Networks A Practical Perspective*. Morgan Kaufmann, 1998.

[18] C.B. Stunkel. The sp2 communication subsystem. Technical report, IBM Thomas J. Watson Research Center, 1994.

[19] T.H. Szymanski and C.V. Hamacher. On the permutation capability of multistage interconnection networks. *IEEE Trans. on Computers*, C-36(7):810–822, 1987.

[20] C.L. Wu and T.Y. Feng. On a class of multistage interconnection networks. *IEEE Trans. on Computers*, C-29(8):694–702, 1980.

[21] Y. Yang, J. Wang, and Y. Pan. Permutation capability of optical multistage interconnection networks. In *Proc. of the 12th International Parallel Processing Symposium and the 9th Symposium on Parallel and Distributed Processing (1998 IPPS/SPDP)*, pages 125–133, 1998.