Gray Codes for Torus and Edge Disjoint Hamiltonian Cycles*

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Abstract

Lee distance Gray codes for k-ary n-cubes and torus networks are presented. Using these Lee distance Gray codes, it is further shown how to directly generate edge disjoint Hamiltonian cycles for a class of k-ary n-cubes, 2-D tori, and hypercubes.

1. Introduction

A multicomputer system consists of multiple nodes that communicate by exchanging messages through an interconnection network. At a minimum, each node normally has one or more processing elements, a local memory, and a communication module. A popular topology for the interconnection network is the *torus*. Also called a *wrap-around mesh* or a *toroidal mesh*, this topology includes the k-ary n-cube which is an n-dimensional torus with the restriction that each dimension is of the same size, k, and the hypercube, which is a k-ary n-cube with $k = 2$; a mesh is a subgraph of a torus.

Several parallel machines, both commercial and experimental, have been designed with a toroidal interconnection network. Included among these machines are the following: Cray T3D and T3E (3D torus) [1], the Mosaic (k-ary n-cube) [2], the iWrap (torus) [3], the Tera Parallel Computer (torus) [4].

Some topological properties of Torus and k-ary n-cubes based on Lee distance are given in [5, 6]. This paper presents some results related to edge disjoint Hamiltonian cycles in these networks. Although the existence of disjoint Hamiltonian cycles in the cross product of various graphs has been discussed in literature [13, 7, 8, 9, 12, 10, 11], a straight forward way of generating such cycles is not clear. Here, first we design Lee distance Gray codes and then using these codes show how to decompose a higher dimension torus to edge disjoint lower dimensional tori - in particular, edge disjoint Hamiltonian cycles.

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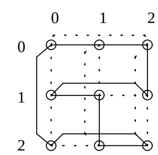


Figure 1. Two disjoint Hamiltonian cycles in $C_3 \times C_3$

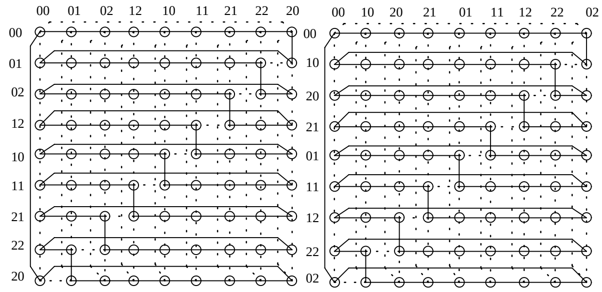
For example, Figure 1 gives two edge disjoint cycles (solid and dotted lines) in $C_3 \times C_3$. Similarly, a $C_3 \times C_3 \times C_3 \times C_3$ is decomposed into two edge disjoint tori of size $C_9 \times C_9$ in Figure 2(a) and Figure 2(b). The solid and dotted lines in these two figures give a total of four disjoint Hamiltonian cycles in $C_3 \times C_3 \times C_3 \times C_3$.

The rest of the paper is organized as follows. Section 2 gives some preliminaries. Lee distance Gray codes (for torus network) are described in Section 3. Section 4 discusses some results on edge disjoint Hamiltonian cycles. Section 5 gives similar results for hypercubes. Section 6 is the conclusion of this paper.

2. Preliminaries

2.1. Lee Distance and Torus

Let $A = a_{n-1}a_{n-2}\cdots a_0$ be a n-dimensional mixed radix vector over Z_K, where $K = k_{n-1} \times k_{n-2} \times \cdots \times k_0$, i.e., all $x_i \in Z_{k_i}$, for $i = 0, 1, \cdots, n-1$. The Lee weight of A in mixed radix notation is defined as $W_L(A) = \sum_{i=0}^{n-1} |a_i|$, where $|a_i| = min(a_i, k_i - a_i)$, for $i = 0, 1, \cdots, n-1$. The Lee distance between two vectors A and B is denoted by $D_L(A, B)$ and is defined



(a) (b)

Figure 2. Decomposition of $C_3 \times C_3 \times C_3 \times C_3$ into two edge disjoint $C_9 \times C_9$ and four edge disjoint Hamiltonian cycles.

to be $W_L(A-B)$. That is, the Lee distance between two vectors is the Lee weight of their digit-wise difference. In other words, $D_L(A,B) = \sum_{i=0}^{n-1} min(a_i - b_i, b_i - a_i)$, where $a_i - b_i$ and $b_i - a_i$ are *mod* k_i operations. For example, when $K = 4 \times 6 \times 3$, $W_L(321) = min(3, 4-3) + min(2, 6-2) + min(1, 3-1) = 1 + 2 + 1 = 4$, and $D_L(123 - 321) = W_L(202) = 3$.

Let $D_H(A,B)$ be the Hamming distance between two vectors A and B, i.e. the number of position in which A and B differ. Then $D_L(A,B) = D_H(A,B)$ when $k_i = 2$ or 3, for all i, and $D_L(A,B) \geq D_H(A,B)$ when $k_i > 3$ for some i. In the rest of the paper, it is assumed that $k_i \geq 3$.

Just as Hamming distance may be used to define the binary hypercube graph, Q_n, the generalized hypercube graph [14], and twisted hypercube graph [15], Lee distance may be used to define the k-ary n-cube graph, C_k^n, and the n-dimensional torus, $T_{k_1,k_2,\cdots,k_n}$.

A k-ary n-cube graph (C_k^n) and an n-dimensional torus ($T_{k_1,k_2,\cdots,k_n}$) are $2n$-regular graphs containing k^n and $k_1 k_2 \cdots k_n$ nodes respectively. Each node in a C_k^n is labeled with a distinct n-digit radix-k vector while each node in a $T_{k_1,k_2,\cdots,k_n}$ is labeled with a distinct n-digit mixed radix vector. In this paper, node labels will be written as $(a_{n-1}a_{n-2}\cdots a_0)$ or as $a_{n-1}a_{n-2}\cdots a_0$ rather than the n-tuples $(a_{n-1}, a_{n-2}, \cdots, a_0)$ when there is no confusion. If u and v are two nodes in the graph, then there is an edge between them *iff* $D_L(u,v) = 1$. From the definition of Lee distance, it can be seen that every node in a C_k^n or a $T_{k_1,k_2,\cdots,k_n}$ shares an edge with two nodes in every dimension, resulting in a regular graph of degree $2n$. In addition, the shortest path between any two vectors, u and v has length $D_L(u,v)$. Note that a C_k^n is an n-dimensional hypercube, Q_n, when $k = 2$.

2.2. Cross Product of Graphs

Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, define the *cross product* of G_1 and G_2, denoted by $G_1 \times G_2$ [16], as the graph $G = (V, E)$, where

$$
\begin{aligned}
V &= \{(u,v) | u \in V_1, v \in V_2\}, \text{ and} \\
E &= \{((u_1,v_1),(u_2,v_2)) | ((u_1,u_2) \in E_1 \text{ and } v_1{=}v_2), \\
&\quad \text{ or } (u_1{=}u_2 \text{ and } (v_1,v_2) \in E_2)\}.
\end{aligned}
$$

A cycle of length k is denoted by C_k, and each node in C_k is labeled with a radix k number, $0 \cdots k-1$. There is an edge between vertices u and v *iff* $D_L(u,v) = 1$.

A k-ary n-cube (C_k^n) and an n-dimensional torus ($T_{k_1,k_2,\cdots,k_n}$) can alternately be defined as the product of cycles as follows.

$$
\begin{aligned}
C_k^n &= \underbrace{C_k \times C_k \times \cdots \times C_k}_{n\ times} = \times_{i=1}^{n} C_k \\
T_{k_1,k_2,\cdots,k_n} &= C_{k_1} \times C_{k_2} \times \cdots \times C_{k_n}
\end{aligned}
$$

The above definition demonstrates a useful topological property of a C_k^n: a C_k^n can be recursively defined in terms of smaller k-ary cubes.

$$
C_k^n = \begin{cases} C_k, & \text{if } n = 1 \\ C_k \times C_k^{n-1}, & \text{if } n > 1 \end{cases}
$$

3. Lee Distance Gray Codes

Many algorithms can be solved efficiently by embedding a Hamiltonian cycle or a Hamiltonian path within torus network. This section addresses the embedding problem by presenting four methods of constructing a Lee distance Gray code. For each method, let $R = (r_{n-1}r_{n-2}\cdots r_0)$ be a number in radix notation, and let $G = (g_{n-1}g_{n-2}\cdots g_0)$ be the Gray code presentation given by f_i, i.e, $G = f_i(R)$. The first three methods are given in [5, 6] and so the proofs are omiited. However, the methods are useful in generating disjoint Hamiltonian cycles and so they are briefly described here.

3.1. Single Radix Codes

First, assume there are n dimensions, each having the same number of processors, k, where $k \geq 3$. Each processor node is labeled with a distinct n-digit, radix k vector $(r_{n-1}r_{n-2}\cdots r_0)$, where $r_i \leq k$ for $0 \leq i \leq n-1$. Two nodes, $A = (a_{n-1}a_{n-2}\cdots a_0)$ and $B = (b_{n-1}b_{n-2}\cdots b_0)$, are adjacent if the Lee distance between them, $D_L(A,B)$, is one.

Two methods are given below for constructing a Gray code based on the assumption of the previous paragraph.

Method 1 [5] $f_1(r_{n-1}r_{n-2}\cdots r_0) = r_{n-1}(r_{n-2} - r_{n-1})\cdots(r_0 - r_1)$

Method 2 [5] This method produces a Hamiltonian cycle if k is even, and a Hamiltonian path if k is odd. Let $\overline{r_i} = k-1-r_i$, and let $g_{n-1} = r_{n-1}$. Then for $i = n-2, \cdots, 0$, if k is even then

$$g_i = \begin{cases} r_i, & \text{if } r_{i+1} \text{ is even} \\ \overline{r_i}, & \text{otherwise} \end{cases}$$

or, if k is odd, let $r' = \sum_{j=i+1}^{n-1} r_j$, and

$$g_i = \begin{cases} r_i, & \text{if } r' \text{ is even} \\ \overline{r_i}, & \text{otherwise} \end{cases}$$

3.2. Mixed Radix Codes

In many cases, however, the number of processors per dimension varies. Let $K = k_{n-1}k_{n-2}\cdots k_0$ be an n-dimensional vector where k_i is the radix of dimension i and $k_i \geq 3$ for $0 \leq i \leq n-1$. In this case, Method 3 gives a Gray code design resulting in a Hamiltonian cycle if k_i is even for at least one value of i. If each k_i is odd, the resulting Gray code produces a Hamiltonian path. Method 4 produces a Hamitonian cycle if all k_i's are odd.

Let each processor node be labeled with a distinct n-digit vector $R = (r_{n-1}r_{n-2}\cdots r_0)$, where $0 \leq r_i \leq k_i - 1$ for $i = 0, 1, \cdots, n-1$. Vector R is said to be in *mixed-radix notation*, and the integer value of R is given by

$$\begin{aligned} I(R) &= r_0 + r_1k_0 + r_2k_0k_1 + \cdots + r_{n-1}k_0k_1\cdots k_{n-2} \\ &= \sum_{i=1}^{n-1}\left(r_i \prod_{j=0}^{i-1} k_j\right) + r_0 \end{aligned}$$

Method 3 [6] Assume that at least one of the k_i's is even. Without loss of generality, assume that the dimensions are ordered so that if k_i is even and k_j is odd, then $i > j$. Let l be the index of the lowest even dimension. That is, the dimensions are ordered as follows.

$$\overbrace{k_{n-1}\cdots k_l}^{even}\ \overbrace{k_{l-1}\cdots k_0}^{odd}$$

Now, letting $\overline{r_i} = k_i - 1 - r_i$, and $r_i' = \sum_{j=i+1}^{l} r_j$, f_3 is defined as follows.

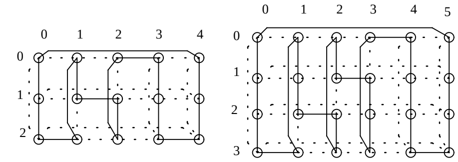
$$g_{n-1} = r_{n-1}, \text{and}$$

for $i = n-2$ downto l:

$$g_i = \begin{cases} r_i, & \text{if } r_{i+1} \text{ is even} \\ \overline{r_i}, & \text{otherwise} \end{cases}$$

for $i = l-1$ downto 0:

$$g_i = \begin{cases} r_i, & \text{if } r_i' \text{ is even} \\ \overline{r_i}, & \text{otherwise} \end{cases}$$



(a) $C_5 \times C_3$ (b) $C_6 \times C_4$

Figure 3. Hamiltonian cycles using Method 4 (The dotted lines are the rest of the edges which form the other edge disjoint Hamiltonian cycle)

Method 4 Assume that k_i is odd for $0 \leq i \leq n-1$, and that the dimensions are ordered such that $k_{n-1} \geq k_{n-2} \geq \cdots \geq k_0$. Also define

$$\overline{r_i} = \begin{cases} r_i, & \text{if } r_{i+1} \text{ is odd} \\ k_i - 1 - r_i, & \text{otherwise} \end{cases}$$

Now, f_4, which produces a Gray code yielding a Hamiltonian cycle, is defined as follows.

$$g_{n-1} = r_{n-1}, \text{and}$$

for $i \leq i \leq n-2$:

$$g_i = \begin{cases} (r_i - r_{i+1})\ mod\ k_i, & \text{if } r_{i+1} < k_i \\ \overline{r_i}, & \text{otherwise} \end{cases}$$

Figure 3(a) shows the Hamiltonian cycles $C_5 \times C_3$ using Method 4 (the rest of the edges form the other Hamiltonian cycle).

Note: When all k_i's, $i = 0, 1, \cdots, n-1$, are even, a similar Gray code described below gives a Hamiltonian cycle. Again assume $k_{n-1} \geq k_{n-2} \geq \cdots \geq k_0$.
Define

$$\overline{r_i} = \begin{cases} r_i, & \text{if } r_{i+1} \text{ is even} \\ k_i - 1 - r_i, & \text{otherwise} \end{cases}$$

Then

$$g_{n-1} = r_{n-1}, \text{and}$$

for $i \leq i \leq n-2$:

$$g_i = \begin{cases} r_i - r_{i+1}, & \text{if } r_{i+1} < k_i \\ \overline{r_i}, & \text{otherwise} \end{cases}$$

Figure 3(b) shows the Hamiltonian cycle in $C_6 \times C_4$ using the above method (Again the rest of the edges form the other Hamiltonian cycle).

Lemma 1 *Method 4 proposed above produces a Lee distance Gray code and hence an Hamiltonian cycle in $T_{k_{n-1},k_{n_2},\cdots,k_0}$, where $k_i \geq 3$, for $i = 0, 1, 2, \cdots, n-1$, and all k_i's are odd (or all k_i's are even).*

Proof: The proof is given when all k_i's, $i = n-1, n-2, \cdots, 0$ are odd. A similar proof is obtained when they are all even.

Case 1:

$$f_4(000\cdots 0) = 0000\cdots 0$$
$$f_4(k_{n-1}-1, k_{n-2}-1, \cdots, k_0-1) = (k_{n-1}-1, \overline{k_{n-2}-1}, \overline{k_{n-3}-1}, \cdots, \overline{k_0-1})$$

But for $i = n-2, n-3, \cdots, 0$, $\overline{k_i - 1} = 0$ because $(k_{i+1} - 1)$'s are all even and $(k_i - 1) < k_{i+1} - 1$. Thus the first and the last words are at a distance of 1.

Case 2: Let X and Y be two consecutive numbers in the mixed radix numbers and let m be the first index from left in which X and Y differ; i.e.,

$$X = \{x_{n-1}x_{n-2}\cdots x_{m+1}\}^* \, x_m \, \{(k_{m-1}-1)(k_{m-2}-1)\cdots(k_0-1)\}^*$$
$$Y = \{x_{n-1}x_{n-2}\cdots x_{m+1}\}^* \, (x_m+1) \, \{00\cdots 0\}^*$$

where the segment marked by a $*$ may or may not exist depending on the value of m. Let $f_4(X) = a_{n-1}a_{n-2}\cdots a_0$ and $f_4(Y) = b_{n-1}b_{n-2}\cdots b_0$. $f_4(X)$ and $f_4(Y)$ are considered in three seqments: (a) between dimension $n-1$ and $m+1$, (b) dimension m, and (c) between dimension $m-1$ and 0. It is shown below that $a_i = b_i$ for $i \neq m$ and that $D_L(a_m, b_m) = 1$. This shows $D_L(f_4(X), f_4(Y)) = 1$.

(a) $[n-1 \geq i \geq m+1]$: For this range, $x_i = y_i$ and so $a_i = b_i$.

(b) $[i = m]$: Also because of (a), either $(a_m = x_m$ and $b_m = x_m + 1)$, or $(a_m = \overline{x_m}$ and $b_m = \overline{x_m+1})$. Further

$$\begin{aligned} \overline{x_m} &= k_m - 1 - x_m \text{ and} \\ \overline{x_m+1} &= k_m - 1 - (x_m+1) \\ &= (k_m - 1 - x_m) - 1 \\ &= \overline{x_m} - 1. \end{aligned}$$

Note that $D_L(x_m, x_{m+1}) = x_m + 1 - x_m = 1$ and that $D_L(\overline{x_m}, \overline{x_m+1}) = \overline{x_m} - (\overline{x_m} - 1) = 1$. Therefore in either case $D_L(a_m, b_m) = 1$.

(c) $[m-1 \geq i \geq 0]$: Since $k_i - 1$ are all even and $k_i - 1 \geq k_{i-1} - 1$, $a_{m-2} = a_{m-3} = \cdots = a_0 = 0$. Further $b_{m-2} = b_{m-3} = \cdots = b_0 = 0$. We show that $a_{m-1} = b_{m-1}$ by considering the following three cases.

case i $[x_m + 1 < k_{m-1}]$: Then $a_{m-1} = k_{m-1} - 1 - x_m$ and $b_{m-1} = 0 - (x_m+1) = k_{m-1} - x_m - 1$. So $a_{m-1} = b_{m-1}$.

case ii $[x_m + 1 = k_{m-1}$ and so $x_{m-1} = k_{m-1} - 1]$: $a_{m-1} = k_{m-1} - 1 - x_m = k_{m-1} - 1 - k_{m-1} - 1 = 0$; $b_{m-1} = 0$ since k_{m-1} is odd. Thus, in this case also $a_{m-1} = b_{m-1}$.

case iii $[x_m + 1 > k_{m-1}]$: If $x_m + 1$ is even(and so x_m is odd) then $a_{m-1} = k_{m-1} - 1$ and $b_{m-1} = \overline{0} = k_{m-1} - 1$; On the other hand, if $\overline{x_{m+1}}$ is odd(and so x_m is even) then $a_m - 1 = \overline{k_{m-1} - 1} = 0$ and $b_{m-1} = 0$. Thus in both cases $a_{m-1} = b_{m-1}$.

□

4. Edge Disjoint Hamiltonian Cycles

When edge disjoint Hamiltonian cycles are used in a communication algorithm, their effectiveness is improved if more than one cycle exists. As mentioned earlier, the existence of disjoint Hamiltonian cycles in the cross product of various graphs has been discussed in the literature [13, 7, 8, 9, 12, 10, 11]; however a straightforward way of generating such disjoint cycles is not clear. This section contains the functions that generate these disjoint cycles for the k-ary n-cubes, and the hypercubes. Because of space limitation, results are given for n, a power of 2. Results for other cases are described in [17] and will be presented in future.

Two Gray codes, G_1 and G_2, over Z_k^n are said to be *independent if* two words, a and b, are adjacent in G_1 (or G_2), then they are not adjacent in G_2 (or G_1). If $k \geq 3$, we can have at most n sets of independent Gray codes; for $k = 2$, this number is $\lfloor \frac{n}{2} \rfloor$.

Theorem 2 *The independent set of Gray codes over Z_k^n is equivalent to the set of edge disjoint Hamiltonian cycles in the graph of k-ary n-cube, C_k^n.*

In the following, $h_i(X) = h_i((x_{n-1}, x_{n-2}, \cdots, x_0); Z_k^n) = (g_{n-1}, g_{n-2}, \cdots, g_0)$ represents the mapping of the radix number form to the corresponding Gray code form, for $i = 0, 1, 2, \cdots, n-1$ (assuming $k \geq 3$), and these mappings give the independent Gray codes. Similarly, $H_0(C_k^n)$, $H_1(C_k^n)$,$\cdots$, $H_{n-1}(C_k^n)$ denote the corresponding Hamiltonian cycles derived by the functions $h_i(X)$, $i = 0, 1, \cdots, n-1$.

For $n = 1$, only one Gray code exists, and so in the rest of the chapter it is assumed that $n \geq 2$ and $k \geq 3$. Further, for $X = (x_{n-1}, x_{n-2}, \cdots, x_0) \in Z_k^n$, let $(X, Z_{k_1}^{n_1})$ represent the n_1 digit long radix-k_1 representation of the radix-k number X. For example, if $X = (2, 4, 4, 3) \in Z_5^4$ then $(X, Z_{25}^2) = (14, 23)$. If there is no confusion, we also assume $X = (X, Z_{k^n}^1)$, i.e., X can be treated an integer in the range 0, 1, $\cdots$, $k^n - 1$.

The following sections show the functions that generate the edge disjoint Hamiltonian cycles for various k-ary n-cubes.

4.1. k-ary 2-cube (i.e., $n = 2$)

Theorem 3 *There are two independent Gray codes in C_k^n and these are generated by the functions h_0 and h_1, where*

$$h_0(X; Z_{k^2}^1) = h_0((x_1, x_0); Z_k^2) = (x_1, (x_0 - x_1) \text{ mod } k)$$
$$h_1(X; Z_{k^2}^1) = h_1((x_1, x_0); Z_k^2) = ((x_0 - x_1) \text{ mod } k, x_1)$$

Proof: h_0 is the same as the function f_1 (Method 1) described in Section 3 of this paper(and also see [5]). Thus this gives a Gray code. h_1 gives two digit sequence which is a permutation of the two digit sequence obtained from h_0. Thus h_1 also gives a Gray code and so an Hamiltonian cycle in k-ary n-cube. We need to prove that the edges are disjoint. In the i-th row, $i = 0, 1, \cdots, k-1$, h_0 uses all edges except the edge $((i, k-1-i), (i, k-i))$ which is the only i-th row edge used by h_1. Similarly, along the j-th column, $j = 0, 1, \cdots, k-1$, h_1 uses all edges except the edge $((j, k-1-j), (j, k-j))$, which is the only j-th column edge used by h_1. Thus the two Hamiltonian cycles are disjoint.

□

The inverse functions of h_0 and h_1 are as follows;

$$h_0^{-1}((g_1, g_0); Z_k^2) = (g_1, (g_0 + g_1) \text{ mod } k)$$
$$h_1^{-1}((g_1, g_0); Z_k^2) = ((g_0 + g_1) \text{ mod } k, g_1)$$

Example 1 *The two Gray code sequences are shown in Figure 1 for ($k = 3$ and $n = 2$). The solid lines represent the first Gray code and the dotted line the second.*

4.2. Two Dimensional Torus, $T_{k^r,k}$

The results given in Section 4.1 can be generalized to two dimensional torus of size $k^r \times k$, for $k \geq 3$ and $r \geq 1$, as shown in the following theorem.

Theorem 4 *There are two independent Gray codes in $T_{k^r,k}$ for $k \geq 3$ and $r \geq 1$ and are generated by the functions h_0 and h_1, where*

$$h_0(x_1, x_0) = (a_1, a_0) = (x_1, (x_0 - x_1) \text{ mod } k)$$
$$h_1(x_1, x_0) = (b_1, b_0)$$
$$= ((x_1(k-1) + x_0) \text{ mod } k^r, x_1 \text{ mod } k)$$

The inverse functions are given by

$$h_0^{-1}(a_1, a_0) = (x_1, x_0) = (a_1, (a_1 + a_0) \text{ mod } k)$$
$$h_1^{-1}(b_1, b_0) = (x_1, x_0)$$
$$= ((b_1 - x_0)(k-1)^{-1} \text{ mod } k^r,$$
$$(b_1 - b_0(k-1)) \text{ mod } k)$$
$$= ((b_1 - x_0)(k-1)^{-1} \text{ mod } k^r, (b_1 + b_0) \text{ mod } k)$$
$$= ((b_1 - ((b_1 + b_0) \text{ mod } k))(k-1)^{-1} \text{ mod } k^r,$$
$$(b_1 + b_0) \text{ mod } k)$$

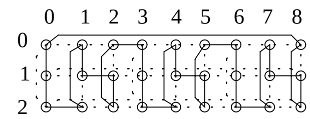


Figure 4. Edge Disjoint Hamiltonian cycles in $T_{9,3}$ produced by h_0 and h_1 of Theorem 5

where $(k-1)^{-1}$ is the multiplicative inverse of $(k-1)$ under mod k^r (Note: For $k \geq 3$, $k-1$ and k^r are relatively prime and so the inverse exists).

Example 2 *Figure 4 shows the two edge disjoint Hamiltonian cycles in $T_{9\times3}$ produced by h_0 (Solid lines) and h_1(dotted lines) of Theorem 4.*

4.3. k-ary n-cube, where $n = 2^r$, and $k \geq 3$

The disjoint Hamiltonian cycles can be recursively constructed as described in the following theorem.

Theorem 5 *The n independent sets of Gray codes are defined by the functions $h_i, i = 0, 1, 2, \cdots, n-1$ as described below. Let $i_1 = \lfloor \frac{2i}{n} \rfloor$ and $h_{i_1}(X; Z_k^n) = h_{i_1}((X_1, X_0); Z_{k^{n/2}}^2) = (Y_1, Y_0)$, i.e.,*

$$(Y_1, Y_0) = h_{i_1}((X_1, X_0); Z_{k^{n/2}}^2)$$
$$= \begin{cases} (X_1, (X_0 - X_1) \text{ mod } k^{n/2}) & \text{for } i_1 = 0, \\ ((X_0 - X_1) \text{ mod } k^{n/2}, X_1) & \text{for } i_1 = 1 \end{cases}$$

Then, the i^{th} function h_i is recursively defined as

$$h_i(X; Z_k^n) = (y_{n-1}, y_{n-2}, \cdots, y_{n/2}, y_{n/2-1}, \cdots, y_0)$$

where,

$$(y_{n-1}, y_{n-2}, \cdots, y_{n/2}) = h_{(i \text{ mod } n/2)}(Y_1; Z_k^{n/2}),$$
$$(y_{n/2-1}, y_{n/2-2}, \cdots, y_0) = h_{(i \text{ mod } n/2)}(Y_0; Z_k^{n/2}).$$

Proof: By induction on n.
Base: when $n = 2$, this theorem is reduced to Theorem 3.
Induction Hypothesis: Assume that there are n independent sets of Gray codes in C_k^n, H_0', H_1', $\cdots$, H_{n-1}' for $n = 2^r$, i.e., $C_k^n = H_0' + H_1' + \cdots + H_{n-1}'$.
Induction Step: Now consider the case for $n' = 2n = 2^{r+1}$. A C_k^n can be decomposed as

$$C_k^{n'} = C_k^n \times C_k^n$$
$$= (H_0' + H_1' + \cdots + H_{n-1}')$$
$$\times (H_0'' + H_1'' + \cdots + H_{n-1}'')$$
$$= (H_0' \times H_0'') + \cdots + (H_{n-1}' \times H_{n-1}'')$$

Here $G_1 + G_2$ indicates the union of two edge disjoint graphs, G_1 and G_2. Now $H_i' \times H_i''$ is a two dimensional torus of size $k^n \times k^n$ and this is edge-disjoint from $H_j' \times H_j''$ for $i \neq j$. From each $H_i' \times H_i''$, two disjoint Hamiltonian cycles can be constructed using Theorem 3. Thus $2n$ disjoint Hamiltonian cycles can be constructed from C_k^n, and the i^{th} Gray code is the i_1^{th} cycle of the component $(H_{i_0}' \times H_{i_0}'')$, where $i_0 = i \bmod \lfloor n/2 \rfloor$ and $i_1 = \lfloor \frac{2i}{n} \rfloor$. □

Example 3 *In Z_4^8, there are 8 independent sets of Gray codes, which are generated by the functions $h_i, i = 0, 1, 2, \cdots, 7$. Let $X = (2,1,1,3,2,3,0,1)$ be a given vector over Z_4^8. We now describe how this vector is mapped under h_3. Since $i = 3$, $i_1 = \lfloor \frac{2\times 3}{8} \rfloor = 0$, and $h_0((2,1,1,3,2,3,0,1); Z_{4^4}^2) = h_0((151,177); Z_{256}^2)$ $= (151, 26) = (Y_1, Y_0)$.*

$$h_3(X; Z_k^n) = (h_{(3 \bmod 4)}(151; Z_4^4), h_{(3 \bmod 4)}(26; Z_4^4))$$
$$= (h_3((2,1,1,3); Z_4^4), h_3((0,1,2,2); Z_4^4)).$$

To find $h_3((2,1,1,3); Z_4^4)$, note that $((2,1,1,3); Z_4^4) = ((9,7), Z_{16}^2)$. In this recursion, $i_1 = \lfloor \frac{2\times 3}{4} \rfloor = 1$ and so $h_1((9,7); Z_{16}^2) = (14, 9)$. Thus

$$h_3((2,1,1,3); Z_4^2)$$
$$= (h_{(3 \bmod 2)}(14; Z_4^2), h_{(3 \bmod 2)}(9; Z_4^2))$$
$$= (h_1((3,2); Z_4^2), h_1((2,1); Z_4^2)) = (3,3,2,2).$$

Similarly it can be shown that $h_3((0,1,2,2); Z_4^4) = (3,2,1,0)$ and thus $h_3((2,1,1,3,2,3,0,1); Z_4^8) = (3,3,3,2,3,2,1,0)$. Further, it can be verified that $h_0(X) = (2,3,3,3,0,1,2,3)$, $h_1(X) = (3,2,3,3,1,0,3,2)$, etc.

Note: In Theorem 5, the functions $h_i(x_{n-1}, x_{n-2}, \cdots, x_0)$, for $i = 1, 2, \cdots, n-1$ can be obtained from $h_0(x_{n-1}, x_{n-2}, \cdots, x_0) = (a_{n-1}, a_{n-2}, \cdots, a_0)$ by a simple permutation of a_i's. Let $i = (i_{n-1}, i_{n-2}, \cdots, i_0)$ in binary. To get $h_i(x_{n-1}, x_{n-2}, \cdots, x_0)$, if $i_j = 1$ for $j = 0, 1, 2, \cdots, n-1$, then permute the least 0-th 2^j bits of h_0 with the next (1-st) 2^j bits, the 2-nd 2^j bits with the 3-rd 2^j bits, etc. For example, suppose

$$\begin{aligned} h_0(X) &= h_{000}(x_7, x_6, \cdots, x_0) \\ &= (a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0) \end{aligned}$$

Then

$$h_1(X) = h_{001}(X) = (\underline{a_6}, \underline{a_7}, \underline{a_4}, \underline{a_5}, \underline{a_2}, \underline{a_3}, \underline{a_0}, \underline{a_1})$$
$$h_2(X) = h_{010}(X) = (\underline{a_5, a_4}, \underline{a_7, a_6}, \underline{a_1, a_0}, \underline{a_3, a_2})$$
$$h_3(X) = h_{011}(X) = (\underline{\underline{a_4}, \underline{a_5}}, \underline{\underline{a_6}, \underline{a_7}}, \underline{\underline{a_0}, \underline{a_1}}, \underline{\underline{a_2}, \underline{a_3}})$$
$$h_4(X) = h_{100}(X) = (\underline{a_3, a_2, a_1, a_0}, \underline{a_7, a_6, a_5, a_4})$$
$$h_5(X) = h_{101}(X) = (\underline{\underline{a_2}, \underline{a_3}, \underline{a_0}, \underline{a_9}}, \underline{\underline{a_6}, \underline{a_7}, \underline{a_4}, \underline{a_5}})$$
$$h_6(X) = h_{110}(X) = (\underline{\underline{a_1, a_0}, \underline{a_3, a_2}}, \underline{\underline{a_5, a_4}, \underline{a_7, a_6}})$$
$$h_7(X) = h_{111}(X) = (\underline{\underline{\underline{a_0}, \underline{a_1}}, \underline{\underline{a_2}, \underline{a_3}}}, \underline{\underline{\underline{a_4}, \underline{a_5}}, \underline{\underline{a_6}, \underline{a_7}}})$$

Thus in the above example,

$$\begin{aligned} h_0(X) &= h_{000}(2,1,1,3,2,3,0,1) \\ &= (2,3,3,3,0,1,2,3) \\ h_1(X) &= h_{001}(X) = (3,2,3,3,1,0,3,2) \\ h_2(X) &= h_{010}(X) = (3,3,2,3,2,3,0,1) \\ h_3(X) &= h_{011}(X) = (3,3,3,2,3,2,1,0) \\ h_4(X) &= h_{100}(X) = (0,1,2,3,2,3,3,3) \\ h_5(X) &= h_{101}(X) = (1,0,3,2,3,2,3,3) \\ h_6(X) &= h_{110}(X) = (2,3,0,1,3,3,2,3) \\ h_7(X) &= h_{111}(X) = (3,2,1,0,3,3,3,2) \end{aligned}$$

Example 4 *Based on Theorem 5, Figure 2 shows the decomposition of $C_3 \times C_3 \times C_3 \times C_3$ into two edge disjoint $T_{9,9}$ and four edge disjoint Hamiltonian cycles (solid and dotted lines).*

5. Hypercubes

The n-dimensional hypercube, Q_n, can be defined as

$$Q_n = Q_{n-1} \times Q_1$$

where Q_1 is a line joining two nodes $(0, 1)$. A two dimensional hypercube, $Q_1 \times Q_1$, is ismorphic to C_4. This can be seen by the mapping $00 \leftrightarrow 0$, $01 \leftrightarrow 1$, $11 \leftrightarrow 2$, and $10 \leftrightarrow 3$. Thus $2n$-dimensional hypercube is ismorphic to 4-ary n-cube(C_4^n).

$$\begin{aligned} Q_{2n} &= \underbrace{Q_1 \times Q_1} \times \underbrace{Q_1 \times Q_1} \times \cdots \times \underbrace{Q_1 \times Q_1} \\ &= Q_2 \times Q_2 \times \cdots \times Q_2 \\ &= C_4 \times C_4 \times \cdots \times C_4 \\ &= C_4^n \end{aligned}$$

Thus, when $n = 2^{2r} = 4^r$, the method described in Section 4.3 can be used to generate the edge disjoint Hamiltonian cycles.

Example 5 *Fig 5 shows two edge disjoint Hamiltonian cycles in Q_4.*

6. Conclusion

The Lee distance Gray codes presented in this paper are shown to be useful in edge disjoint decomposition of higher dimensional torus to lower dimensional tori - in particular edge disjoint Hamiltonian cycles. The mapping functions, which map a node in radix number system to the corresponding Gray codewords are simple. In fact, for k-ary n-cube, it is required to calculate only one mapping function,

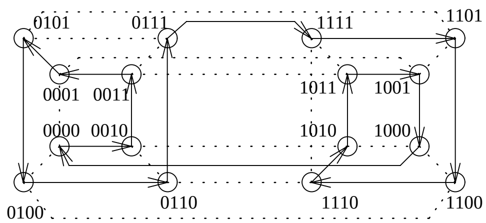


Figure 5. Two edge disjoint Hamiltonian cycles in Q_4

which gives the first Gray code and so the first Hamiltonian cycle. The other $n - 1$ mappings are some simple permutation of this function. Because of the space limitations, the results are given for n, the dimension of torus, to be a power of 2. For other cases, similar results will be given in the future.

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