Permutation Routing in All-Optical Product Networks

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Abstract. In this paper we study permutation routing techniques for all-optical networks. Firstly, we show some lower bounds on the number of wavelengths needed for implementing any permutation on an all-optical network in terms of bisection of the network. Secondly, we study permutation routing on product networks by giving a lower bound on the number of wavelengths needed, and presenting permutation routing algorithms for the wavelength non-conversion and conversion models, respectively. Finally, we investigate permutation routing on a cube-connected-cycles network by showing that the number of wavelengths needed for implementing any permutation in one round is $\lfloor 2 \log n \rfloor$, which improves on a previously known general result for bounded degree graphs by a factor of $O(\log^3 n)$ for this special case.

1 Introduction

In this paper we address designing routing algorithms for an emerging new generation of networks known as *all-optical network* [1,10,16,24]. This kind of network offers the possibility of interconnecting hundreds to thousands of users, covering local to wide area, and providing capacities exceeding substantially those a conventional network can provide. The network promises data transmission rates several orders of magnitudes higher than current electronic networks. The key to high speed in the network is to maintain the signal in optical form rather than electronic form. The high bandwidth of the fiber-optic links is utilized through *Wavelength-Division Multiplexing* (WDM) technology which supports the propagation of multiple laser beams through a single fiber-optic link provided that each laser beam uses a distinct optical wavelength. Each laser beam, viewed as a carrier of signal, provides transmission rates of 2.5 to 10 Gbps, thus the significance of WDM in very high speed networking is paramount. The major applications of the network are in video conferencing, scientific visualization and real-time medical imaging, supercomputing and distributed computing [24,25]. A comprehensive overview of the physical theory and applications of this technology can be found in the books by Green [10] and McAulay [16]. Some studies [1,18,23] model the all-optical network as an *undirected* graph, and thus the routing paths for implementing requests on the network are also undirected. However, it has since become apparent that optical amplifiers placed on fiber-optic links are directed devices. Following [5,9,17], we will model the all-optical network as a *directed symmetric* graph, and each routing path on it is a directed path.

The Model. An all-optical network consists of vertices (nodes, stations, processors, etc), interconnected by point-to-point fiber-optic links. Each fiber-optic link supports a given number of wavelengths. The vertex may be occupied either by terminals, switches, or both. *Terminals* send and receive signals. *Switches* direct the input signals to one or more of the output links. Several types of optical switches exist. The *elementary switch* is capable of directing the signals coming along each of its input links to one or more of the output ones. The elementary switch cannot, however, differentiate between different wavelengths coming along the same link. Rather, the entire signal is directed to the same output(s) [6,1,23]. The *generalized switch*, on the other hand, is capable of switching incoming signals based on their wavelengths [1,23]. Using acousto-optic filters, the switch splits the incoming signals to different streams associated with various wavelengths, and may direct them to separate outputs. In both cases, on any given link different messages must use different wavelengths. Unless otherwise specified, we adopt generalized switches for our routing algorithms. Besides there being a difference in the use of switches, there is a difference regarding the wavelength assignment for routing paths. In most cases, we only allow the assignment of a unique wavelength to each routing path. We call this WDM model the *Wavelength Non-Conversion Model*. If we allow each routing path to be assigned different wavelengths for its different segments, we call it a *Wavelength Conversion Model* (called the *λ-routing model* in [7]).

Previous Related Work. Optical routing in an arbitrary undirected network G was considered by Raghavan and Upfal [23]. They proved an $\Omega(1/\beta^2)$ lower bound on the number of wavelengths needed to implement any permutation in one round, where β is the edge expansion of G (which is defined later). They also presented an algorithm which implements any permutation on bounded degree graphs using $O(\log^2 n/\log^2 \lambda)$ wavelengths within one round with high probability, where n is the number of nodes in G and λ is the second largest eigenvalue (in absolute value) of the transition matrix of the standard random walk on G. For degree d arrays, they presented an algorithm with an $O(dn^{1/d}/\log n)$ worst case performance. Aumann and Rabani [5] presented a near optimal implementation algorithm for bounded degree networks in one round. Their algorithm needs $O(\log^2 n/\beta^2)$ wavelengths. For any bounded dimension array, any given number of wavelengths, and an instance I, Aumann and Rabani [5] suggested an algorithm which realizes I using at most $O(\log n \log |I| T_{opt}(I))$ rounds, where $T_{opt}(I)$ is the minimum number of rounds necessary to implement I. Later, Kleinberg and Tardos [13] obtained an improved bound of $O(\log n)$ on the approximation of the number of rounds required to implement I. Recently, Rabani [22] further improved the result in [13] to $O(poly\ \log\log n T_{opt}(I))$. Pankaj [18] considered the permutation routing issue in hypercube based networks. For hypercube, shuffleexchange, and De Bruijn networks, he showed that routing can be achieved with $O(\log^2 n)$ wavelengths. Aggrawal et al [1] showed that $O(\log n)$ wavelengths are sufficient for the routing in this case. Aumannn and Rabani [5] demonstrated that routing on hypercubes can be finished with a constant number of wavelengths. Gu and Tamaki [11] further showed that two wavelengths for directed symmetric hypercubes and eight wavelengths for undirected hypercubes are sufficient. Pankaj [18,19] proved an $\Omega(\log n)$ lower bound on the number of wavelengths needed for routing a permutation in one round on a bounded degree network. Barry and Humblet [6,7] gave bounds for routing in passive (switchless) and λ-networks. An almost matching upper bound is presented later in [1]. Peiris and Sasaki [20] considered bounds for elementary switches. The connection between packet routing and optical routing is discussed in [1]. The integral multicommodity flow problem related to optical routing has been discussed in [5,3,4]. A comprehensive survey for optical routing is in [9].

Our Results. In this paper we first present some lower bounds on the number of wavelengths needed for implementing any permutation on all-optical networks in terms of the bisections of the networks. We then consider the product networks, which can be decomposed into a *direct product* of two networks. Many well known networks such as hypercubes, meshes, tori, etc, are product networks. We first show a lower bound on the number of wavelengths needed for implementing any permutation on product networks in one round, then present permutation routing algorithms for product networks, based on the wavelength non-conversion and conversion models respectively. Finally we study the permutation issue on cube-connected-cycles networks by implementing any permutation in one round using only $\lfloor 2\log n \rfloor$ wavelengths. This improves on a general result for bounded degree networks in [5] by a factor of $O(\log^3 n)$ for this special case.

2 Preliminaries

We define some basic concepts involved in this paper.

Definition 1. An all-optical network can be modeled as a *directed symmetric* graph $G(V, E)$ with $|V| = n$ and $|E| = m$, where for each pair of vertices u and v, if a directed edge $\langle u, v\rangle \in E$, then $\langle v, u\rangle \in E$ too.

Definition 2. A *request* is an ordered pair of vertices (u, v) in G which corresponds to a message to be sent from u to v. u is the *source* of v and v is the *destination* of u.

Definition 3. An *instance* I is a collection of requests. If $I = \{(i, \pi(i)) \mid i \in V\}$, where π is a permutation of vertices in G.

Definition 4. Let $P(x, y)$ denote a directed path in G from x to y. A *routing* for an instance I is a set of directed paths $\mathcal{R} = \{P(x, y) \mid (x, y) \in I\}$. An instance I is *implemented* by assigning wavelengths to the routing paths and setting the switches accordingly.

Definition 5. The *conflict graph*, associated with a permutation routing

$$\mathcal{R} = \{P(i, \pi(i)) \mid i \in V,\ i \text{ is a source and } \pi(i) \text{ is the destination of } i\}$$

on a directed or undirected graph $G(V, E)$, is an undirected graph $G_{\mathcal{R},\pi} = (V', E')$, where each directed (undirected) routing path in $\mathcal{R}$ is a vertex of V' and there is an edge in E' if the two corresponding paths in $\mathcal{R}$ share at least a common directed (or undirected) edge of G.

Definition 6. The *edge-expansion* $\beta(G)$ of $G(V, E)$ is the minimum, over all subsets S of vertices, $|S| \leq n/2$, of the ratio of the number of edges leaving S to the size of S $(\subset V)$.

Definition 7. A *bisection* of a graph $G(V, E)$ is defined as follows: Let P be a partition which partitions V into two disjoint subsets V_1 and V_2 such that $|V_1| = \lceil |V|/2 \rceil$ and $|V_2| = \lfloor |V|/2 \rfloor$. The *bisection problem* is to find such a partition (V_1, V_2) that $|C|$ is minimized, where $C = \{(i, j) \mid i \in V_2,\ j \in V_1, \text{ and } (i, j) \in E\}$. Let $c(G) = |C|$. The bisection concept for undirected graphs can be extended to directed graphs. For the directed version, define $C = \{\langle i, j\rangle \mid i \in V_2,\ j \in V_1, \text{ and } \langle i, j\rangle \in E\}$.

Definition 8. The *congestion* of a permutation π on $G(V, E)$ is defined as follows. Let $\mathcal{R}$ be a set of routing paths for π on G. Define $C(e, \mathcal{R}) = \{P(i, \pi(i)) \mid i \in V,\ e \in P(i, \pi(i)), \text{ and } P(i, \pi(i)) \in \mathcal{R}\}$. Then, the congestion problem for π on G is to find a routing $\mathcal{R}$ such that $\max_{e \in E}\{|C(e, \mathcal{R})|\}$ is minimized. Denote by

$$congest(G, \pi) = \min_{\mathcal{R}} \max_{e \in E}\{|C(e, \mathcal{R})|\}.$$

Let Π be the set of all permutations on G. Then, the congestion of G, $congest(G)$, is defined as

$$congest(G) = \max_{\pi \in \Pi}\{congest(G, \pi)\}.$$

3 Lower Bounds

Theorem 9. *For any all-optical network $G(V, E)$, let $c(G)$ be the number of edges in a bisection of G, $congest(G)$ be the congestion of G, and $w_{\min}$ be the number of wavelengths needed to implement any permutation on G in one round. Then*

$$w_{\min} \geq congest(G) \ \geq\ \frac{n-1}{2c(G)}.$$

Proof. Following the congestion definition and the optical routing rule that different signals through a single fiber-optic link must be assigned different wavelengths, it is easy to see that $w_{\min} \geq congest(G)$.

Let (V_1, V_2) be a bisection of G and $|V_1| \geq |V_2|$. Assume that there is a permutation π which permutes all vertices in V_2 to V_1. Then the congestion of G for π is $congest(G, \pi) \geq \lfloor n/2 \rfloor / c(G) \geq \frac{n-1}{2c(G)}$, Since $congest(G) \geq congest(G, \pi)$, the theorem then follows. □

Theorem 9 always holds no matter whether the WDM model is the wavelength conversion model or not. From this theorem we have the following corollaries.

Corollary 10. *In a chain L_n of n vertices, the number of wavelengths for implementing any permutation on it in one round is at least $\lfloor n/2 \rfloor$.*

Corollary 11. *In a ring R_n of n vertices, the number of wavelengths for implementing any permutation on it in one round is at least $\frac{n-1}{4}$.*

Corollary 12. *Let M be $l \times h$ mesh with $n = hl$ vertices. Suppose $l \leq h$. The number of wavelengths for implementing any permutation on M in one round is at least $\frac{\sqrt{n}}{2} - 1$.*

4 Product Networks

Define the *direct product* of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, denoted by $G \times H$, as follows. The vertex set of $G \times H$ is the Cartesian product $V_1 \times V_2$. There is an edge between vertex (v_1, v_2) and vertex (v_1', v_2') in $G \times H$ when $v_1 = v_1'$ and $(v_2, v_2') \in E_2$, or $v_2 = v_2'$ and $(v_1, v_1') \in E_1$. G and H are called the *factors* of $G \times H$. Notice that $G \times H$ consists of $|V_2|$ copies of G connected by $|V_1|$ copies of H, arranged in a grid-like fashion. Each copy of H is called a *row* and each copy of G is called a *column*. An edge in $G \times H$ between (v_1, u) and (v_2, u) is called a *G-edge* if $(v_1, v_2) \in E_1$, and an edge of $G \times H$ between (v, u_1) and (v, u_2) is called an *H-edge* if $(u_1, u_2) \in E_2$. Now let $W = G \times H$ and π be a permutation on W. Assume the factors G and H of W are equipped with routing algorithms. We are interested in designing a routing algorithm for W by using the routing algorithms for its factors as *subroutines*.

4.1 Lower bounds on the number of wavelengths

Lemma 13. *Let $G \times H$ be a directed symmetric product network. Let $c(X)$ be the number of the edges in a bisection of X and $V(X)$ be the vertex set of X. Suppose that d_G and d_H are the maximum in-degrees (out-degrees) of G and H. Then we have the following lower bounds for $c(G \times H)$:*

1. $\min\{|V(H)|c(G), |V(G)|c(H)\}$ *if both $|V(G)|$ and $|V(H)|$ are even;*
2. $\min\{|V(G)|c(H) + \lfloor |V(G)|/2 \rfloor d_H + c(G), |V(H)|c(G)\}$ *if $|V(G)|$ is even and $|V(H)|$ is odd;*
3. $\min\{|V(H)|c(G) + \lfloor |V(H)|/2 \rfloor d_G + c(H), |V(G)|c(H)\}$ *if $|V(G)|$ is odd and $|V(H)|$ is even;*
4. $\min\{|V(G)|c(H) + \lfloor |V(G)|/2 \rfloor d_H + c(G), |V(G)|c(G) + \lfloor |V(G)|/2 \rfloor d_G + c(H)\}$ *otherwise (both $|V(G)|$ and $|V(H)|$ are odd).*

Proof. We first consider the case where $p = |V(G)|$ and $q = |V(H)|$ are even. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of G. Suppose the bisection of G is $(V_1(G), V_2(G))$, where $V_1(G) = \{v_1, v_2, \ldots, v_{p/2}\}$ and $V_2(G) = \{v_{p/2+1}, \ldots, v_p\}$.

We replace each vertex of G by graph H, and make the corresponding connection between two vertices in different copies of H. This results in the graph W. Now we partition the vertex set of W into two disjoint subsets of equal size according to the bisection of G. Thus, the number of edges of W in this partition is $|V(H)|c(G)$. Since W is a symmetric network, there is another partition which satisfies the above partition condition. The number of edges in this latter partition is $|V(G)|c(H)$. Since $c(G \times H)$ is a partition that has the minimum number of edges, $c(G \times H) \leq \min\{|V(H)|c(G), |V(G)|c(H)\}$.

We then consider the case where $p = |V(G)|$ is even and $q = |V(H)|$ is odd. We use only G-edges of W for the partition. Then, the number of edges of W in this partition is $|V(H)|c(G)$ by the above discussion. In the rest we consider using both H- and G-edges for the bisection of W. Let $V(H) = \{u_1, u_2, \ldots u_q\}$. Suppose that a bisection of H is $(V_1(H), V_2(H))$, where $V_1(H) = \{u_1, u_2, \ldots, u_{\lceil q/2 \rceil}\}$ and $V_2(H) = \{u_{\lceil q/2 \rceil + 1}, \ldots, u_q\}$. It is clear that $|V_1(H)| - |V_2(H)| = 1$ because q is odd. Now we give another partition $(V_1'(H), V_2'(H))$ of H, where $V_1'(H) = V_2(H) \cup \{v\}$ and $V_2'(H) = V_1(H) - \{v\}$. The number of H-edges in this new partition is no more than $c(H) + d_H(v) \leq c(H) + d_H$. We replace each vertex of H by graph G, and make the corresponding connection between two vertices in different copies of G. This leads to the graph W. Partition the vertex set of W into two disjoint subsets such that: the size difference is at most one according to a bisection of H; and the vertices in the copy of G corresponding to v of H are partitioned into two disjoint subsets by a bisection of G. Thus, the number of edges of W in this partition is no more than $\lceil |V(G)|/2 \rceil c(H) + \lfloor |V(G)|/2 \rfloor (c(H) + d_H(v)) + c(G) \leq |V(G)|c(H) + \lfloor |V(G)|/2 \rfloor d_H + c(G)$. Therefore $c(G \times H) \leq \min\{|V(H)|c(G), |V(G)|c(H) + \lfloor |V(G)|/2 \rfloor d_H + c(G)\}$. The other two cases can be shown similarly, omitted. □

Having Lemma 13, the following lower bounds on the number of wavelengths are derived easily.

Theorem 14. *Let $W = G \times H$ be a directed symmetric product network. Let $c(X)$ be the number of edges in a bisection of X. Suppose that d_G and d_H are the maximum in-degrees (out-degrees) of G and H. Then, a lower bound for the minimum number of wavelengths $w_{\min}(W)$ for implementing any permutation on W in one round is:*

1. $\max\{\frac{|V(G)|}{2c(G)} - 1, \frac{|V(H)|}{2c(H)} - 1\}$ *if both $|V(G)|$ and $|V(H)|$ are even;*
2. $\max\{\frac{|V(H)|}{2c(H)+d_H+2c(G)/|V(G)|} - 1, \frac{|V(G)|}{2c(G)} - 1\}$ *if $|V(G)|$ is even and $|V(H)|$ is odd;*
3. $\max\{\frac{|V(G)|}{2c(G)+d_G+2c(H)/|V(H)|} - 1, \frac{|V(H)|}{2c(H)} - 1\}$ *if $|V(G)|$ is odd and $|V(H)|$ is even;*
4. $\max\{\frac{|V(G)|}{2c(G)+d_G+2c(H)/|V(H)|} - 1, \frac{|V(H)|}{2c(H)+d_H+2c(G)/|V(G)|} - 1\}$ *otherwise (both $|V(G)|$ and $|V(H)|$ are odd).*

Proof. From Theorem 9, it is clear that $w_{\min}(W) \geq \frac{|V(G)||V(H)|}{2c(G \times H)} - 1$. Use of the lower bounds for $c(G \times H)$ from Lemma 13 in this inequality gives the theorem. □

4.2 A routing algorithm on the packet-passing model

The permutation routing on a direct product network W has been addressed for the packet-passing model [2,8]. Baumslag and Annextein [8] presented an efficient algorithm for permutation routing on that model which consists of the following three phases. 1. Route some set of permutations on the copies of G; 2. Route some set of permutations on the copies of H; 3. Route some set of permutations on the copies of G. Since the product network is a symmetric network, the above three phases can be applied alternatively, i.e., by first routing on H, followed by G, and followed by H.

Now, consider the following naive routing method. First route each source in a column to its destination row in the column. Then route each source in a row to its destination column in the row. This fails to be an edge congestion-free algorithm because there may be several sources in the same column that have their destinations in a single row (causing congestion at the intersection of the column and the row). By using an initial extra phase, the above congestion problem can be solved. That is, we first "rearrange" each column so that, after the rearrangement, each row consists of a set of sources whose destinations are all in distinct columns. After this, a permutation of each row is required to get each source to its correct column after the rearrangement. Once all of the sources are in their correct columns, a final permutation of each column suffices to get each source to its correct destination. This final phase is indeed a permutation since all destinations of sources are distinct. Thus, the aim of the first phase is to find a set of sources P_R, once per column, such that every source in P_R has its destination in a distinct column for each row R.

To this end, a bipartite graph $G_B(X, Y, E_B)$ is constructed as follows. Let X and Y represent the set of columns of W. There is an edge between $x_i \in X$ and $y_j \in Y$ for each source in column i whose destination is in column j. Since π is a permutation, it follows that G_B is a regular, bipartite multigraph. Thus, G_B can be decomposed into a set of edge disjoint perfect matchings. Note that the sources involved in a single perfect matching all have their destinations in distinct columns. Therefore, for each row R, use all of the sources that correspond to a single perfect matching for the set P_R. Each of these sets P_R, is "lifted" to row R during the first phase of the algorithm. Since each source is involved in precisely one perfect matching, the mapping of sources in a column during the first phase is indeed a permutation of the column.

4.3 Routings on the wavelength conversion model

The algorithm in Section 4.2 can be expressed in a different way. That is, π is decomposed into three permutations σ_i, $i = 1, 2, 3$, where σ_1 and σ_3 are permutations in columns of W and σ_2 is a permutation in rows of W. For example, let v be a source in W at the position of row i_1 and column j_1, and $\pi(v) = u$ be the destination of v at the position row i_2 and column j_2. Suppose v is "lifted" to the position of row i_1' and column j_1 in the first phase. Then,

$\sigma_1(i_1, j_1) = (i'_1, j_1)$, followed by the second phase of $\sigma_2\sigma_1(i_1, j_1) = \sigma_2(i'_1, j_1) = (i'_1, j_2)$, and followed by the third phase of $\sigma_3\sigma_2\sigma_1(i_1, j_1) = \sigma(i'_1, j_2) = (i_2, j_2)$.

We now present a permutation routing algorithm on W for π on the wavelength conversion model, based on the above observation. We start with the following major theorem.

Theorem 15. *Given permutation routing algorithms for networks G and H, there is a routing algorithm for the product network $G \times H$. The number of wavelengths for any permutation in one round is at most* $\max\{2w(G), w(H)\}$ *if $w(G) \leq w(H)$; or* $\max\{2w(H), w(G)\}$ *otherwise, where $w(X)$ represents the number of wavelengths needed to implement any permutation on network X in one round.*

Proof. The routing algorithm by Baumslag and Annexstein is for the packet-passing model, which is a totally different model from our WDM model. Hence, some modifications to their algorithm are necessary. As we can see, implementing the permutation routing on W for π can be decomposed into three permutations σ_i, $1 \leq i \leq 3$.

Without loss of generality, we assume $w(G) \leq w(H)$. Following the three phases of the above algorithm, the permutation σ_1 can be implemented with $w(G)$ wavelengths (in other words, all routing paths can be colored with $w(G)$ colors); the permutation σ_2 can be implemented with $w(H)$ wavelengths (all routing paths can be colored with $w(H)$ colors, and the colors for σ_1 can be re-used here); the permutation σ_3 can be implemented with $w(G)$ wavelengths (all routing paths can be colored with $w(G)$ colors, and the colors for σ_1 cannot be used here). Now, for a request $(i, \pi(i))$, let (u_1, v_1) be the position of i in W and (u_2, v_2) be the position of $\pi(i)$ in W. Then, the routing path L_i for request $(i, \pi(i))$ consists of three routing segments $L_{i,1}$, $L_{i,2}$ and $L_{i,3}$ which correspond to σ_1, σ_2, and σ_3, where $L_{i,1}$ only contains the vertices in column v_1 of W and consists of G-edges; $L_{i,2}$ only contains the vertices in row u'_1 of W and consists of H-edges, assuming that $\sigma_1(i)$ is at row u'_1 and column v_1 in W; and $L_{i,3}$ only contains the vertices in column v_2 of W, and G-edges.

Any path L_i can further be made a simple path L'_i by deleting the cycles it contains. Let $L'_{i,j}$ be the corresponding segment of $L_{i,j}$, $1 \leq j \leq 3$. L'_i then is assigned three different wavelengths for each of the segments of L'_i, which are the wavelengths for $L_{i,j}$ originally, $j = 1, 2, 3$. That is, each L'_i for a request $(i, \pi(i))$ can be implemented with at most three wavelengths. But, the wavelengths used for $L_{i,1}$ cannot be used for $L_{i,3}$. So, at least $2w(G)$ wavelengths are needed. In summary, implementing any permutation π on W in one round can be done with $\max\{2w(G), w(H)\}$ wavelengths. □

Following Theorem 15, the following corollaries can be derived directly.

Corollary 16. *There is an algorithm for implementing any permutation in a directed symmetric hypercube H_q of $n = 2^q$ vertices with 2 wavelengths.*

Proof. Since $H_q = K_2 \times H_{q-1}$ and $w(K_2) = 1$, by Theorem 15, $w(H_q) = \max\{2w(K_2), w(H_{q-1})\}$. Furthermore, $H_{q-1} = K_2 \times H_{q-2}$, and it is easy to show that $w(H_q) = 2$ by induction on q. □

Corollary 17. *For any directed symmetric $l \times h$ mesh M with $l \leq h$ and $n = lh$, there is an algorithm for implementing any permutation on M in one round. The number of wavelengths used by this algorithm is at most* $\max\{l, \lfloor h/2 \rfloor\}$.

Proof. Assume $l \leq h$ and $M = L_l \times L_h$, where L_i represents a chain of i vertices. Obviously $w(L_i) = \lfloor i/2 \rfloor$. By Theorem 15, $w(M) = \max\{2w(L_l), w(L_h)\} \leq \max\{l, \lfloor h/2 \rfloor\}$. □

Corollary 18. *There is an algorithm for implementing any permutation in a directed symmetric $\sqrt{n/2} \times \sqrt{2n}$ mesh with n vertices in one round. The number of wavelengths used by this algorithm is at most $\sqrt{n/2}$ which is almost optimal.*

Proof. Let M be the $\sqrt{n/2} \times \sqrt{2n}$ mesh. By Corollary 17, $w(M) = \max\{2w(L_{\sqrt{n/2}}), w(L_{\sqrt{2n}})\} = \sqrt{n/2}$, and by Corollary 12, the lower bound of the number of wavelengths for any permutation on M is $\Omega(\sqrt{n})$, so, this bound is almost tight. □

4.4 Routings on the wavelength non-conversion model

We now study the permutation routing on W for the wavelength non-conversion model in which each routing path is assigned a unique wavelength. We only consider the case of $w(G) \leq w(H)$. The case of $w(H) \leq w(G)$ is similar, omitted.

Following the proof of Theorem 15, a routing path L'_i for every $i \in V(W)$ is obtained, and L'_i can be further divided into three segments $L'_{i,k}$, $k = 1, 2, 3$. Each of the segments has been assigned a wavelength (a color). Let γ_j be the color (wavelength) of $L'_{i,k}$. Then $(\gamma_1, \gamma_2, \gamma_3)$ is the ordered color tuple of L'_i. We treat $(\gamma_1, \gamma_2, \gamma_3)$ as a coordinate point in a 3-dimensional Cartesian coordinate system. Assume that each coordinate point in the system has been assigned a unique label, i.e., L'_i is assigned the wavelength numbered by the label of $(\gamma_1, \gamma_2, \gamma_3)$. Then, the total number of coordinate points for all routing paths on W in a permutation is $w(G) \times w(H) \times w(G) = w(G)^2 w(H)$.

Now, consider two routing paths L'_i and L'_j, $i \neq j$. Let L'_i be colored with $(\gamma_1, \gamma_2, \gamma_3)$ and L'_j be colored with $(\beta_1, \beta_2, \beta_3)$. If there exists a k such that $\gamma_k \neq \beta_k$, $1 \leq k \leq 3$, then the wavelengths assigned for L'_i and L'_j are different. Otherwise, if $\gamma_k = \beta_k$ for all k, $1 \leq k \leq 3$, then L'_i and L'_j are assigned the same wavelength. However, this wavelength assignment may not be incorrect because there still exist some routing paths sharing common edges which are assigned the same wavelength. We illustrate this situation by an example (see Figure 1). Assume that the given permutation is π. Consider two routing paths L'_i and L'_j, where L'_i starts from i, goes through y', $\pi(j)$, x, x', and ends at $\pi(i)$, L'_j starts from j, goes through x', $\pi(i)$, y, y', and ends at $\pi(j)$. Clearly L'_i and L'_j share two common segments which are from y' to $\pi(j)$ and from x to $\pi(i)$ even though they have the same color tuple. So, they cannot be assigned the same wavelength on the WDM model. To cope with this situation, the following approach is applied. Let $l_{\max}(G)$ be the number of edges in the longest routing path on G for any permutation. Define $\mathcal{R}_l = \{L'_i \mid L'_i \text{ is labeled by } l \text{ in the coordinate system}\}$.

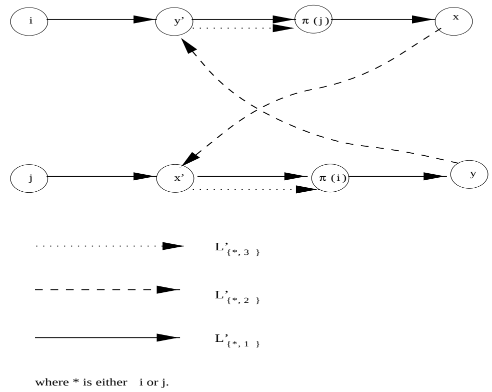

Fig. 1. An example.

Obviously the set of routing paths for π is $\mathcal{R} = \bigcup_{l=1}^{w(G)^2 w(H)} \mathcal{R}_l$. For each $\mathcal{R}_l$, an auxiliary graph $G_l = (V_l, E_l)$ that is a subgraph of the conflict graph on W, is constructed as follows. Every vertex in V_l corresponds to an element in $\mathcal{R}_l$. There is an edge between two vertices if the two routing paths share at least one common edge. Then we have

Lemma 19. *Let $G_l = (V_l, E_l)$ be defined as above, then the maximum degree of G_l is $l_{\max}(G)$.*

Proof. Let L'_i and L'_j be the corresponding routing paths of two vertices in G_l. We know that L'_k consists of three segments $L'_{k,1}$, $L'_{k,2}$, and $L'_{k,3}$, $k = i$ or $k = j$. By the definition, $L'_{i,p}$ and $L'_{j,p}$ are edge disjoint for all p, $1 \le p \le 3$. But $L'_{i,1}$ and $L'_{j,3}$ (similarly $L'_{j,1}$ and $L'_{i,3}$) may not be edge disjoint, see Fig. 1. Since the number of edges in any routing path is no greater than $l_{\max}(G)$, there are at most $l_{\max}(G)$ other routing paths sharing common edges with L'_i. Therefore, the maximum degree of G_l is $l_{\max}(G)$. □

By Lemma 19, all vertices in G_l can be colored with $l_{\max}(G) + 1$ colors such that the adjacent vertices are colored with different colors. This coloring can be done in polynomial time by a greedy approach. This means, for each $\mathcal{R}_l$, we can assign $l_{\max}(G) + 1$ wavelengths for the routing paths in it such that those paths sharing common edges are assigned different wavelengths. There are $w(G)^2 w(H)$ different $\mathcal{R}_l$s. Therefore, the total number of wavelengths required for any permutation is $(l_{\max}(G) + 1)w(G)^2 w(H)$, which is formally described as follows.

Theorem 20. *Given permutation routing algorithms for networks G and H, there is a permutation routing algorithm for the product network $W = G \times H$. The number of wavelengths for any permutation on W in one round is $(l_{\max}(G) + 1)w(G)^2 w(H)$ if $w(G) \le w(H)$; or $(l_{\max}(H) + 1)w(H)^2 w(G)$ otherwise, where*

$w(X)$ represents the number of wavelengths needed to implement any permutation on network X in one round, and $l_{\max}(X)$ is the number of edges of the longest routing path on X for any permutation.

Let $|V(G)| = p$, $|V(H)| = q$, and $n = pq$. Assume that both p and q are functions of n. If both $w(H)$ and $w(G)$ are linear functions of the vertex sizes of G and H, for example, $w(G) = ap$ and $w(H) = bq$ where a and b are constants with $0 < a, b < 1$. Without loss of generality, we further assume that $w(G) \leq w(H)$. Then, it needs $(l_{\max}(G) + 1)w(G)^2 w(H) = (l_{\max}(G) + 1)(a^2 b)pn = cn^{1+\alpha} > n$ wavelengths where $p = n^{\alpha}$. As a matter of the fact, any permutation can be implemented on an arbitrary all-optical network in one round if n wavelengths are available. To cope with this case, we present another permutation routing algorithm. We start with the following perfect matching lemma.

Lemma 21. *[15] Let $G_B(X, Y, E)$ be a bipartite graph such that for every subset S of X, we have $|N(S)| \geq |S|$, where $N(S)$ is the subset of Y that are adjacent to vertices in S. Then G_B has a perfect matching of size* $\min\{|X|, |Y|\}$.

Suppose $p \leq q$. Our idea comes from Youssef [26]. That is, at each time, we select p sources and their destinations such that these sources belong to distinct rows and their destinations belong to distinct columns. Such sources can be found using perfect matching in a bipartite graph $G_B = (X, Y, E)$ where X is the set of rows and Y is the set of columns. There is an edge connecting $x \in X$ and $y \in Y$ if there is a source in row x whose destination is in column y. Clearly G_B is a bipartite multigraph, the degree of every vertex of G_B in X is q, and the degree of every vertex of G_B in Y is p. Since any subset $S \subseteq X$, $|N(S)| \geq |S|$. Thus, there is a perfect matching in G_B by Lemma 21. By deleting this matching, we can find the next perfect matching in the remaining graph, and so on. As a result, G_B is decomposed into q edge disjoint perfect matchings. Since the routing paths in a perfect matching are edge disjoint, they can be assigned the same wavelength. So, we have

Lemma 22. *Given a directed symmetric network $G \times H$ with $|V(G)| = p$, $|V(H)| = q$ and $p \leq q$, there is an algorithm for implementing any permutation on $G \times H$ in one round if q wavelengths are available.*

If we allow $\max\{w(G), w(H)\}$ wavelengths to assign every fiber-optic link of $G \times H$, we have

Theorem 23. *Let $G \times H$ be a directed symmetric network. On the wavelength non-conversion model, if there are permutation algorithms for implementing any permutation on G and H in one round with $w(G)$ and $w(H)$ wavelengths respectively, then there is a permutation algorithm for implementing any permutation on $G \times H$ in $\frac{|V(H)|}{\max\{w(H), w(G)\}}$ $(\leq 2c(H) + 1)$ rounds with $\max\{w(H), w(G)\}$ wavelengths if $|V(G)| \leq |V(H)|$, or in $\frac{|V(G)|}{\max\{w(H), w(G)\}}$ $(\leq 2c(G) + 1)$ rounds with $\max\{w(H), w(G)\}$ wavelengths, where $w(G)$ and $w(H)$ are the linear functions of their sizes and $c(X)$ is the number of edges in a bisection of X.*

Proof. We only consider the case $|V(G)| \leq |V(H)|$. The analogous case $|V(H)| < |V(G)|$ is omitted. By Theorem 9, $w(H) \geq \frac{|V(H)|-1}{2c(H)}$. So, $|V(H)| \leq 2w(H)c(H)+1$. According to Lemma 22, in order to implement any permutation on $G \times H$ with $\max\{w(H), w(G)\}$ wavelengths, the number of rounds needed is at most

$$\frac{|V(H)|}{\max\{w(H), w(G)\}} \leq \frac{2w(H)c(H)+1}{\max\{w(H), w(G)\}}.$$

That is, the number of rounds is at most $|V(H)|/w(H) \leq \frac{2w(H)c(H)+1}{w(H)} \leq 2c(H)+1$ if $w(G) \leq w(H)$; or $|V(H)|/w(G) \leq \frac{2w(H)c(H)+1}{w(G)} \leq 2c(H)+1$ otherwise. □

5 Routings for Specific All-Optical Networks

In this section we study permutation routings on some specific all-optical networks. Before we proceed, we introduce Lemma 24 by Gu and Tamaki [11].

Lemma 24. *Let H_q be a directed symmetric hypercube. Then any permutation on it can be implemented in one round with two wavelengths.*

We now consider the well known *cube-connected-cycles* (CCC) network [21,14], which is constructed from the r-dimensional hypercube by replacing each vertex of the the hypercube with a cycle of r vertices in the CCC. The resulting graph has $n = r2^r$ vertices with degree 3. By modifying the labeling scheme of the hypercube, we can represent each vertex by a pair $\langle w, i \rangle$ where i $(1 \leq i \leq r)$ is the position of the vertex within its cycle and w (any r-bit binary string) is the label of the vertex in the hypercube that corresponds to the cycle. Then two vertices $\langle w, i \rangle$ and $\langle w', i' \rangle$ are linked by an edge in the CCC if and only if either (1) $w = w'$ and $i - i' \equiv \pm 1 \bmod r$, or (2) $i = i'$ and w differs from w' in precisely the ith bit. Edges of the first type are called *cycle edges*, while edges of the second type are referred to as *hypercube edges*. For the CCC, we have the following theorem.

Theorem 25. *Let $G(V, E)$ be a directed symmetric CCC network defined as above. Then any permutation on G can be implemented either in $\lfloor \log n \rfloor$ rounds using 2 wavelengths, or in one round using $\lfloor 2 \log n \rfloor$ wavelengths on the wavelength non-conversion model.*

Proof. The basic idea is to choose a vertex from each cycle as a source such that no two sources have their destinations in the same cycle. Route these chosen sources to their destinations on the supergraph, which can be implemented in one round using 2 wavelengths by Lemma 24. Thus, any permutation on the CCC can be implemented in r rounds. Since $r = \log n - \log r \leq \lfloor \log n \rfloor$, $\lfloor \log n \rfloor$ rounds suffices.

It remains to show how to choose the sources for each round such that none of the destinations of two sources are in the same cycle. The approach proceeds as follows. First we construct a bipartite graph $G(X, Y, E_{XY})$ where X and Y

represent the sets of 2^r cycles of G. For $v \in X$ and $u \in Y$, if a source in v will route its message to a destination in u, then an undirected edge (u, v) is added to E_{XY}. Clearly, $G(X, Y, E_{XY})$ is a regular bipartite multigraph with degree r. By Hall's theorem, $G(X, Y, E_{XY})$ can be decomposed into r edge disjoint perfect matchings, and each perfect matching can be found in polynomial time. The corresponding routing paths for each perfect matching can be implemented using two wavelengths by Lemma 24. Therefore, any permutation in the CCC can be implemented with either $\lfloor \log n \rfloor$ rounds using 2 wavelengths, or one round using $2 \times r \leq \lfloor 2 \log n \rfloor$ wavelengths. □

Note that the CCC is a regular network with out-degree (in-degree) 3. Aumann and Rabani [5] showed that $O(\log^2 n / \beta^2(G))$ wavelengths suffice for implementing any permutation on a network G of constant degree in one round, where $\beta(G)$ is the edge-expansion of G. The CCC is a network of degree three, and $\beta(\text{CCC}) \leq \frac{2^{r-1}}{2^{r-1} r} = 1/r$. For this special network, our permutation routing algorithm implements any permutation in one round with $2 \log n$ wavelengths, which improves the number of wavelengths used in [5] by a factor of $O(\log^3 n)$.

Next we generalize the CCC further to the following network $G(V, E)$ with $n = 2^r \times s$ vertices satisfying $s \geq r$. This network is a hypercube supergraph of 2^r supervertices, and each supervertex is a cycle of s vertices. Clearly CCC is a special case of this generalized network with $s = r$. Now we see that the number of wavelengths needed for implementing any permutation on G depends on the number of vertices of degree three. For example, if $2^r = \sqrt{n}$, then $s = \sqrt{n}$. The network G contains $2^r \times r = \frac{\sqrt{n} \log n}{2}$ vertices of degree three, and $n - \frac{\sqrt{n} \log n}{2}$ vertices of degree two. Thus, implementing any permutation on G using the above approach requires either $\sqrt{n}$ rounds if there are 2 wavelengths available, or one round if there are $2\sqrt{n}$ wavelengths available. Thus, we have

Lemma 26. *Let $G(V, E)$ be a directed symmetric, generalized CCC network defined as above, with $s \geq r$. Then any permutation on G can be implemented either in s rounds using 2 wavelengths, or in one round using $2s$ wavelengths on the wavelength non-conversion model. The number of vertices of degree three in G is $r2^r$, and the number of vertices of degree two is $n - r2^r$.*

6 Conclusions

In this paper we have shown some lower bounds on the number of wavelengths needed for implementing any permutation on all-optical networks in terms of bisection. We have also shown a lower bound on the number of wavelengths required on the product networks, and presented permutation routing algorithms for it, based on the wavelength non-conversion and conversion models respectively. We finally considered the permutation issue on a cube-connected-cycles network. The number of wavelengths needed for implementing any permutation on this network in one round is $\lfloor 2 \log n \rfloor$, which improves on a general result for bounded degree networks in [5] by a factor of $O(\log^3 n)$ for this special case.

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