Hyper-Butterfly Network: A Scalable Optimally Fault Tolerant Architecture

Wei Shi and Pradip K Srimani
Department of Computer Science
Colorado State University
Ft. Collins, CO 80523

Abstract

Bounded degree networks like deBruijn graphs or wrapped butterfly networks are very important from VLSI implementation point of view as well as for applications where the computing nodes in the interconnection networks can have only a fixed number of I/O ports. One basic drawback of these networks is that they cannot provide a desired level of fault tolerance because of the bounded degree of the nodes. On the other hand, networks like hypercube (where degree of a node grows with the size of a network) can provide the desired fault tolerance but the design of a node becomes problematic for large networks. In their attempt to combine the best of the both worlds, authors in [1] proposed hyper-deBruijn networks that have many additional features of logarithmic diameter, partitionability, embedding etc. But, hyper-deBruijn networks are not regular, are not optimally fault tolerant and the optimal routing is relatively complex. Our purpose in the present paper is to extend the concepts used in [1] to propose a new family of scalable network graphs that retain all the good features of hyper-deBruijn networks as well as are regular and maximally fault tolerant; the optimal point to point routing algorithm is very simple.

1 Introduction

It is desirable to investigate if we can combine the low I/O port requirements of bounded degree networks with the advantage [2] of having a desired level of fault tolerance. Authors in [1] have proposed a new family of graphs, called hyper-deBruijn graphs, that meets both these requirements and provides other features like logarithmic diameter, optimal routing algorithms, scalability and partitionability and ability to emulate most of existing architectures. Hyper-deBruijn graphs have two shortcomings: (1) they are not regular (nor they are Cayley graphs; hence optimal routing algorithms and VLSI implementations are significantly complicated; (2) the fault tolerance is lower than the degree of the vast majority of nodes in the graph (due to the existence of few nodes of smaller degrees). Our purpose in the present paper is to propose a new family of composite graphs (combination of hypercubes and wrapped butterfly graphs) which are regular Cayley graphs, have logarithmic diameter, extremely simple optimal routing and are maximally fault tolerant. A quick comparison between the hypercubes, wrapped butterfly, hyper-deBruijn and hyper-butterfly graphs is presented in Figure 1. The proposed family of graphs is interesting from graph theoretic point of view and more importantly, offers a better and more attractive alternative to hyper-deBruijn graphs for VLSI implementation in terms of regularity and greater fault tolerance.

2 Hyper-Butterfly graph

2.1 Hypercube Graph H_n and Butterfly Graph B_n

A hypercube H_n, of order n, is defined to be regular symmetric graph $G = (V, E)$ where V is the set of 2^n vertices, each representing a distinct n-bit binary number and E is the set of symmetric edges such that two nodes are connected by an edge iff the Hamming distance between the two nodes is 1 i.e., the number of positions where the bits differ in the binary labels of the two nodes is 1. It is known that the number of edges in H_n is $n2^{n-1}$ and the diameter of H_n is given by $\mathcal{D}(H_n) = n$. It is also known that the vertex connectivity of H_n is n i.e., there are exactly n vertex disjoint paths between any two nodes in H_n. A hypercube H_n can be defined for N vertices only when $N = 2^n$.

A wrapped butterfly network, denoted by B_n, is defined [3] as follows: a vertex is represented as $\langle z_1 z_2 \cdots z_n, \ell \rangle$, where $z_1 z_2 \cdots z_n$ is a n-bit binary number and ℓ is an integer, $0 \leq \ell \leq n-1$; two vertices $\langle z_1 z_2 \cdots z_n, \ell \rangle$ and $\langle z'_1 z'_2 \cdots z'_n, \ell' \rangle$ are connected by a bidirectional edge iff $(\ell' = \ell + 1 (\bmod n)) \wedge (z'_i = z_i, \ \forall i \vee z'_i =$

Parameter	Hypercube	Butterfly	Hyper-deBruijn	Hyper-Butterfly
Nodes	2^{m+n}	$(m+n)\times 2^{m+n}$	2^{m+n}	$n\times 2^{m+n}$
Edges	$(m+n)2^{m+n-1}$	$(m+n)2^{m+n+1}$	2^{m+n+1}	$n2^{m+n+1}$
Regular	Yes	Yes	No	Yes
Degree	$m+n$	4	$m+4$	$m+4$
Diameter	$m+n$	$\lceil\frac{3n}{2}\rceil$	$m+n$	$m+\lceil\frac{3n}{2}\rceil$
Fault-tolerance	$m+n$	4	$m+2$	$m+4$
Embeddings				
Cycles	Even Cycles	Even Cycles	Pancyclic	Even Cycles
Mesh	Yes	No	Yes	Yes
Binary Tree	$T(m+n-1)$	$T(m+n+1)$	$T(m+n-1)$	$T(m+n-1)$
Mesh of Trees	Yes	Yes	Yes	Yes

Figure 1: Hyper-deBruijn $HD(m,n)$ and Hyper-Butterfly $HB(m,n)$ Graphs Compared

$z_i,\ \forall i,$ except for $i=\ell'$). Recently, the same butterfly topology (with wrap around) B_n is redefined in [4] as a graph on $n\times 2^n$ vertices for any integer n, $n\geq 3$; each vertex is represented by a cyclic permutation of n symbols in lexicographic order where each symbol may be present in either uncomplemented or complemented form. Let t_k, $1\leq k\leq n$ denote the k-th symbol in the set of n symbols. Since each node is some cyclic permutation of the n symbols in lexicographic order, then if $a_1a_2\cdots a_n$ denotes the label of an arbitrary node and $a_1=t_k^*$ for some integer k, then for all i, $2\leq i\leq n$, we have $a_i=t^*_{((k+i) \bmod n)+1}$. The edges of B_n are defined by the following four generators in the graph:

$$\begin{aligned}
g(a_1a_2\cdots a_n) &= a_2a_3\cdots a_na_1\\
f(a_1a_2\cdots a_n) &= a_2a_3\cdots a_n\bar{a}_1\\
g^{-1}(a_1a_2\cdots a_n) &= a_na_1\cdots a_{n-1}\\
f^{-1}(a_1a_2\cdots a_n) &= \bar{a}_na_1\cdots a_{n-1}
\end{aligned}$$

Remark 1 *B_n is a symmetric (undirected) regular graph of degree 4, has $n\times 2^n$ nodes and $n\times 2^{n+1}$ edges. B_n has a logarithmic diameter $\mathcal{D}(B_n)=\lfloor\frac{3n}{2}\rfloor$ and B_n has a vertex connectivity 4, i.e., for any pair of nodes there exist 4 node disjoint paths between them. B_n has many other interesting properties; see [4] for details.*

Remark 2 *It is easy to see the equivalence (isomorphism) between the two interpretations of the wrapped butterfly network graph. We highlight the key features below.*

Each node in B_n has one of the n possible cyclic permutations of the n symbols (disregarding the complementation of the symbols).

Definition 1 *The* **permutation index (PI)** *of the identity node I is zero and that of any other node is defined to be an integer x if the node's permutation is obtained from I by x left shifts.*

For example, each of the nodes $\{abc,\ \bar{a}bc,\ a\bar{b}c,\ ab\bar{c}\}$ has a PI of zero while $PI(\bar{b}\bar{c}a)=1$ and $PI(cab)=2$ and so on.

Definition 2 *For any node $v = a_1a_2\cdots a_n$ in B_n, the* **complementation index (CI)** *of node v is defined as $CI(v)=\sum_{j=1}^{n}w_i*2^{j-1}$ where w_i is 0 if a_i is an uncomplemented symbol and 1 if a_i is a complemented symbol.*

2.2 Hyper-Butterfly Graph $HB(m,n)$

Consider two undirected graphs $G=(V_G,E_G)$ and $H=(V_H,E_H)$; the product graph $G\times H$ has node set $V_G\times V_H$. Let u and v be any two nodes in G, and let x and y be any two nodes in H; then, $(\langle u,x\rangle,\langle v,y\rangle)$ is an edge of $G\times H$ iff either (1) (u,v) is an edge of G and $x=y$, or (2) (x,y) is an edge of H and $u=v$.

Definition 3 *A* **Hyper Butterfly** *graph $HB(m,n)$, of order (dimension) $(m+n)$ is defined as the product graph of a hypercube H_m of dimension m and a butterfly B_n of dimension n.*

Each node in $HB(m,n)$ is assigned a label $(x_{m-1}\ldots x_0,t_{n-1}\cdots t_0)$ where each x_i is a binary bit and each t_j is a symbol, either complemented or uncomplimented; the symbols t_j, $0\leq j\leq n-1$ are all distinct (as in the definition of the butterfly graph). We refer to $x_{m-1}\cdots x_0$ as the *hypercube-part-label* and $t_{n-1}\cdots t_0$ as the *butterfly-part-label* of any node in $HB(m,n)$. The edges of the $HB(m,n)$ graph are defined by the following $m+4$ generators:

$$\begin{aligned}
&\forall i,\ 0\leq i<n,\ h_i(x_{m-1},\cdots,x_0,t_{n-1}\cdots,t_0)=\\
&\quad x_{m-1},\cdots,x_{i+1},\bar{x}_i,x_{i-1},\cdots,x_0,t_{n-1}\cdots t_0,\\
&g(x_{m-1},\cdots,x_0,t_{n-1},\cdots,t_0)=\\
&\quad x_{m-1},\cdots,x_0,t_{n-2},\cdots,t_0t_{n-1}
\end{aligned}$$

$$f(x_{m-1},\cdots,x_0,t_{n-1},\cdots,t_0) = x_{m-1},\cdots,x_0,t_{n-2},\cdots,t_0\bar{t}_{n-1}$$
$$g^{-1}(x_{m-1},\cdots,x_0,t_{n-1},\cdots,t_0) = x_{m-1},\cdots,x_0,t_0t_{n-1},\cdots,t_1$$
$$f^{-1}(x_{m-1},\cdots,x_0,t_{n-1},\cdots,t_0) = x_{m-1},\cdots,x_0,\bar{t}_0t_{n-1},\cdots,t_1$$

Remark 3

- *The set of $m+4$ generators of the graph $HB(m,n)$, $\Omega = \{h_i, 0 \le i < m, f, g, f^{-1}, g^{-1}\}$ is closed under inverse; in particular $h^{(i)}$ for all i is its own inverse, g is inverse of g^{-1} and f is inverse of f^{-1}; thus the edges in $HB(m,n)$ are bidirectional.*

- *For an arbitrary n, $n > 2$, for any arbitrary node v of the graph $HB(m,n)$, $\delta(v) \ne v$ where $\delta \in \Omega = \{h_i, 0 \le i < m, f, g, f^{-1}, g^{-1}\}$; also, for any two $\delta_1, \delta_2 \in \Omega$, $\delta_1(v) \ne \delta_2(v)$.*

Theorem 1 *Hyper butterfly graph $HB(m,n)$ is a Cayley graph of degree $m+4$.*

Definition 4 *(i) The m edges generated by the generators h_i are called* **hypercube edges** *and the 4 edges generated by either of the generators g, f, g^{-1}, f^{-1} are called* **butterfly edges***.*

(ii) Any arbitrary node $v = (h,b) \in HB(m,n)$ has m **hypercube neighbors** *$\{(h^{(i)}), 1 \le i \le m\}$ (reached from v by the m hypercube edges) and has 4* **butterfly neighbors** *$\{(h, b^{(j)}), 1 \le j \le 4\}$ (reached from v by the 4 butterfly edges).*

Remark 4 *Along any hypercube edge, only the hypercube-part-label of a node changes, and along any butterfly edges, only the butterfly-part-label changes.*

Theorem 2 *For any m and n, $n \ge 3$, the graph $HB(m,n)$ (1) is a symmetric (undirected) regular graph of degree $m+4$; (2) has $n \times 2^{m+n}$ vertices; and (3) has $(m+4) \times n \times 2^{m+n-1}$ edges.*

Proof : (1) follows from Remark 3. (2) follows from the definition that $HB(m,n)$ is the product graph of $H(m) \times B(n)$. Since $HB(m,n)$ has the node-set $V_{H(m)} \times V_{B(n)}$, the number of nodes in $HB(m,n)$ is $2^m \times n \times 2^n$ which is equal to $n \times 2^{m+n}$. (3) follows from (1) and (2). $\square$

3 Shortest Routing and Diameter in $HB(m,n)$

Shortest (optimal) point to point routing in $HB(m,n)$ is extremely simple and elegant. Recall that each node u in $HB(m,n)$ has a two-part label (h,b) where h denotes the hypercube-part-label and b denotes the butterfly-part-label. Consider all nodes in $HB(m,n)$ with the same (but arbitrary) butterfly-part-label; there are 2^m such nodes and they form a hypercube H_m of dimension m. Similarly, consider all nodes in $HB(m,n)$ with the same (but arbitrary) hypercube-part-label; there are $n \times 2^n$ such nodes and they form a butterfly B_n of dimension n. We can formalize this observation as the following remark.

Remark 5 *A Hyper-deBruijn graph $HB(m,n)$ can be decomposed into $n \times 2^n$ many mutually disjoint hypercubes H_m, each of the H_m nodes having the same butterfly-part-label. Similarly, an $HB(m,n)$ can be decomposed into 2^m many mutually disjoint butterfly B_n, each of the B_n nodes having the same hypercube-part-label.*

Shortest (optimal) point to point routing in $HB(m,n)$ is then obvious in light of the above discussion. Consider two arbitrary nodes (h,b) and (h',b') in $HB(m,n)$. The shortest route can be established as follows:

- Go from node (h,b) to (h',b) using the shortest routing scheme in a hypercube [5].

- Go from node (h',b) to node (h',b') using the shortest routing scheme in butterfly graphs [4].

Remark 6 *The fact that the above algorithm actually computes the shortest path follows from the correctness of the shortest routing schemes in hypercube and butterfly graphs.*

Remark 7 *Since $HB(m,n)$ is a Cayley graph, it is vertex symmetric [6], i.e., we can always view the distance between any two arbitrary nodes as the distance between the source node and the identity node $(\overbrace{00\cdots0}^{m\ bits},\overbrace{t_1t_2\cdots t_n}^{n\ symbols})$ by suitably renaming the symbols representing the node labels.*

Theorem 3 *The diameter of $HB(m,n)$ is $m + \lceil \frac{3n}{2} \rceil$.*

Proof : From the construction of the shortest routing algorithm, the length of the optimal path between two arbitrary point in $HB(m,n)$ is upper bounded the sum of the diameter m of the hypercube H_m and the diameter $\lceil \frac{3n}{2} \rceil$ of the butterfly B_n. Now, consider the identity node $(00\cdots0, t_1t_2\cdots t_n)$ and the node $v = (11\cdots1, \bar{t}_{\lceil \frac{n}{2}+1}\cdots\bar{t}_n\bar{t}_1\cdots\bar{t}_{\lceil \frac{3n}{2} \rceil})$. The distance between these two nodes is evidently $m + \lceil \frac{3n}{2} \rceil$ and the claim follows. $\square$

Remark 8 *The distance between two arbitrary nodes (h, b) and (h', b') in $HB(m, n)$ is given by the sum of the distances between two nodes h and h' in a hypercube H_m and the distance between two nodes b and b' in a butterfly graph B_n.*

4 Embeddings in $HB(m, n)$

In this section we briefly state our results on embedding without any proofs (for lack of space). A cycle with k nodes, denoted by $C(k)$, is a graph whose vertices comprise the set $\{1, 2, \cdots, k\}$ and edge set is defined by $E = \{(i, j) | i = (i + 1) \bmod k\}$.

Remark 9 *The hypercube $H(m)$ of dimension m has a cycle of length k as a subgraph for an even k and $4 \le k \le 2^m$ [5] and the butterfly B_n of dimension n has a cycle of length $kn + 2k'$ where k and k' are positive integers and $k + k' \le 2^n$ [7].*

A wrap-around mesh (or a 2-dimensional mesh with wrap-around connections or a torus), denoted by $C(n_1) \times C(n_2)$ or $M(n_1, n_2)$, is a direct product of the two cycles $C(n_1)$ and $C(n_2)$. Also note that for two given graphs, G containing a cycle $C(n_1)$ and H containing a cycle $C(n_2)$, the product graph $G \times H$ contains the wrap-around mesh $C(n_1) \times C(n_2)$ or $M(n_1, n_2)$.

Lemma 1 *A mesh $M(n_1, n_2)$ contains all even cycles of length k, $4 \le k \le n_1 \times n_2$.*

Lemma 2 *The hyper-butterfly graph $HB(m, n)$ has a cycle of length k, when k is even and $4 \le k \le n \times 2^{(m+n)}$.*

Lemma 3 *The butterfly graph B_n has the complete binary tree $T(n + 1)$ as a subgraph.*

Lemma 4 *A mesh of trees $MT(2^p, 2^q)$ is a subgraph of the product graph $T(p + 1) \times T(q + 1)$.*

Theorem 4 *The hyper butterfly graph $HB(m, n)$ embeds a mesh of trees $MT(2^p, 2^q)$, for $1 \le p \le m - 2,\ 1 \le q \le n$.*

5 Fault Tolerance of $HB(m, n)$

The node fault tolerance of an undirected graph is measured by the vertex connectivity of the graph. A graph G is said to have a vertex connectivity ξ if the graph G remains connected when ar arbitrary set of less than ξ nodes are faulty (i.e., in the fault free graph there are ξ many node disjoint paths between any two arbitrary nodes). Obviously, the vertex connectivity of a graph G cannot exceed the minimum degree of a node in G. A graph is called **maximally fault tolerant** if vertex connectivity of the graph equals the minimum degree of a node. We know that the vertex connectivity of a hypercube H_m is m [5]; since H_m is m-regular, the hypercube graphs are maximally fault tolerant. It is also known [4] that the vertex connectivity of the butterfly graph B_n is 4; since B_n is 4-regular, the butterfly graphs are also maximally fault tolerant. Our purpose in this section is to show that the proposed graph $HB(m, n)$ has a vertex connectivity of $m+4$ and hence these graphs are maximally fault tolerant.

Theorem 5 *Between any two arbitrary nodes (h, b) and (h', b') in $HB(m, n)$, there exist $(m + 4)$ node disjoint paths.*

Proof: Consider two arbitrary nodes $u = (h, b)$ and $v = (h', b')$ in $HB(m, n)$. We need to consider three cases:

Case 1:$[h \ne h' \wedge b = b']$, i.e., the source and the destination nodes have the same butterfly-part-label but have different different hypercube-part-labels. Consider the hypercube (H_m, b) (the hypercube formed by all the nodes in $HB(m, n)$ with same butterfly part label b); we have $u, v \in (H_m, b)$ and hence there are m vertex disjoint paths between the nodes u and v in the hypercube (H_m, b) (note that each node on any of those paths has the same butterfly part label b). Now, consider the four butterfly neighbors (reached by butterfly edges) of the node (h, b) —- $\{(h, b^{(i)})\}$, $1 \le i \le 4$. Each node $(h, b^{(i)})$, $1 \le i \le 4$ can reach the node $(h', b^{(i)})$ using nodes in the hypercube $(H_m, b^{(i)})$ and one can reach the node (h', b') from $(h', b^{(i)})$ by traversing one edge. Note that the nodes on these paths are mutually disjoint and these 4 paths are also pairwise node disjoint from the earlier m paths. Thus, we have $m + 4$ node disjoint paths between nodes u and v. Note that the length of any of the first m paths is upper bounded by $m + 2$ and that of any of the last 4 paths is upper bounded by $\lceil \frac{3n}{2} \rceil + 2$.

Case 2:$[h = h' \wedge b \ne b']$, i.e., the source and the destination nodes have the same hypercube-part-labels but have different butterfly-part-labels. Consider the butterfly (h, B_n) (the butterfly formed by all the nodes in $HB(m, n)$ with same hypercube-part-label h); we have $u, v \in (h, B_n)$ and hence there are 4 vertex disjoint paths between the nodes u and v in the butterfly (h, B_n) (note that each node on any of those paths has the same hypercube part lebel h). Now, consider the m hypercube neighbors (reached by hypercube edges) of the node (h, b) —- $\{(h^{(i)}, b)\}$, $1 \le i \le m$. Each node $(h^{(i)}, b)$, $1 \le i \le m$, can reach the node $(h^{(i)}, b')$ using nodes in the butterfly $(h^{(i)}, B_n)$ and one can reach the node (h', b') from $(h^{(i)}, b')$ by traversing one edge. Note that the nodes on these paths are mutually disjoint (butterfly-part-labels of nodes in each path are different) and these m paths are also

Parameter	Hyper-butterfly(3,8)	Hyper-deBruijn(3,11)	Hyper-deBruijn(6,8)
Nodes	16384	16384	16384
Edges	57344	57344	81920
Degree	7	7	10
Diameter	15	14	14
Fault-tolerance	6	4	7
Embeddings			
Cycles	Even cycle	Pancyclic	Pancyclic
2-dimensional Mesh	8×2048	8×2048	64×256
Binary Tree	$T(10)$	$T(13)$	$T(13)$
Mesh of Trees	$MT(2^1, 2^8)$	$MT(2^1, 2^{10})$	$MT(2^4, 2^6)$

Figure 2: Comparison between Hyper-deBruijn and Hyper-Butterfly Graphs

pairwise node disjoint from the earlier 4 paths. Thus, we have $m+4$ node disjoint paths between nodes u and v.

Case 3:$[h \neq h' \wedge b \neq b']$, i.e., the source and the destination nodes have different hypercube-part-labels and different butterfly-part-labels. Consider the m hypercube neighbors $(h^{(i)}, b)$, $1 \leq i \leq n$ (of node u) and 4 butterfly neighbors $(h, b^{(j)})$, $1 \leq j \leq 4$. For each neighbor $(h^{(i)}, b)$, use nodes in the butterfly subgraph $(h^{(i)}, B_n)$ to reach the node $(h^{(i)}, b')$ and then use the nodes in the hypercube subgraph (H_m, b') to reach the node $v = (h', b')$; for each neighbor $(h, b^{(j)})$, use nodes in the hypercube subgraph $(H_m, b^{(j)})$ to reach the node $(h', b^{(j)})$ and then use the nodes in the butterfly subgraph (h', B_n) to reach the node $v = (h', b')$. It is easy to see that the nodes on each of these paths are then mutually disjoint. □

Corollary 1 *The hyper-butterfly graph $HB(m,n)$ of dimension $m+n$ has a vertex connectivity of $m+4$ and hence it is maximally fault tolerant.*

Remark 10 *The constructive proof for the theorem on vertex connectivity (Theorem 5) readily suggests an optimal routing scheme in the network in the presence of maximal number of allowable faults (such that the system is not disconnected).*

6 Conclusion

The class of networks proposed in this paper retain all of the desirable features of the networks of [1] and at the same time provides two additional features: regularity and maximal (optimal) fault tolerance. We compare hyper-deBruijn graph and hyper-butterfly graph in Figure 2; we compare a $HB(3,8)$ with a $HD(3,11)$ and $HD(6,8)$ which accommodate the same number of nodes. The proposed graph seems to be an attractive alternative to the hyper-deBruijn networks [1] for designing multiprocessor networks. We have also recently developed an asymptotically optimal broadcasting algorithm for this proposed network and obtained some interesting results about the VLSI implementation of the proposed topology. We intend to report them soon in a future paper.

References

[1] E. Ganesan and D. K. Pradhan. The hyper-deBruijn networks: scalable versatile architecture. *IEEE Transactions on Parallel and Distributed Systems*, 4(9):962–978, September 1993.

[2] C. Chen, D. P. Agrawal, and J. R. Burke. dBCube: a new class of hierarchical multiprocessor interconnection networks with area efficient layout. *IEEE Transactions on Parallel and Distributed Systems*, 4(12):1332–1344, December 1993.

[3] F. T. Leighton. *Introductions to Parallel Algorithms and Architectures: Arrays, Trees and Hypercubes.* Morgan Kaufman, 1992.

[4] P. Vadapalli and P. K. Srimani. A new family of Cayley graph interconnection networks of constant degree four. *IEEE Transactions on Parallel and Distributed Systems*, 7(1), January 1996.

[5] Y. Saad and M. H. Shultz. Topological properties of hypercubes. *IEEE Transactions on Computers*, 37(7):867–872, July 1988.

[6] S. B. Akers and B. Krishnamurthy. A group-theoretic model for symmetric interconnection networks. *IEEE Transactions on Computers*, 38(4):555–566, April 1989.

[7] P. Vadapalli and P. K. Srimani. Fault tolerant ring embedding in tetravalent Cayley network graphs. *Journal of Circuits, Systems and Computers*, 6(5):527–536, 1996.